EC8451
ELECTROMAGNETIC FIELDS

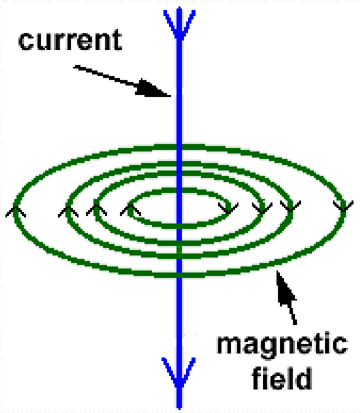
Unit I
Introduction

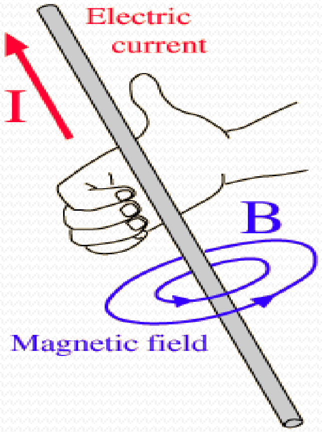
Electromagnetism

- Electricity and magnetism are different facets of *electromagnetism*
  - a moving electric charge produces magnetic fields
  - changing magnetic fields move electric charges
- This connection first elucidated by Faraday, Maxwell
- Einstein saw electricity and magnetism as frame-dependent facets of *unified* electromagnetic force

Magnetic fields from electricity

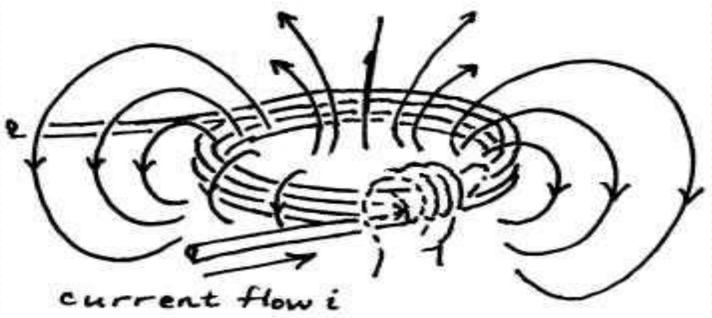
- A static distribution of charges produces an electric field
- Charges in *motion* (an electrical current) produce a magnetic field
  - electric current is an example of charges (electrons) in motion



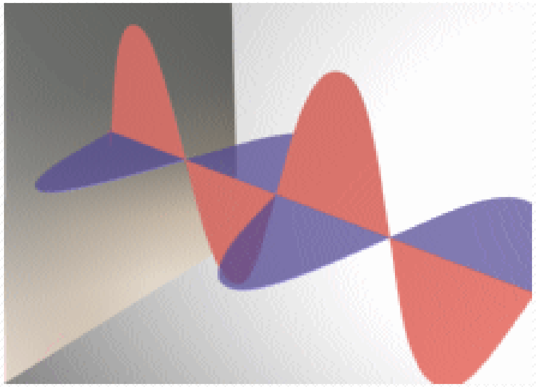


Electromagnets

- Arranging wire in a coil and running a current through produces a magnetic field that looks a lot like a bar magnet
  - called an electromagnet
  - putting a real magnet inside, can shove the magnet back and forth depending on current direction: called a solenoid



Electromagnetic Radiation

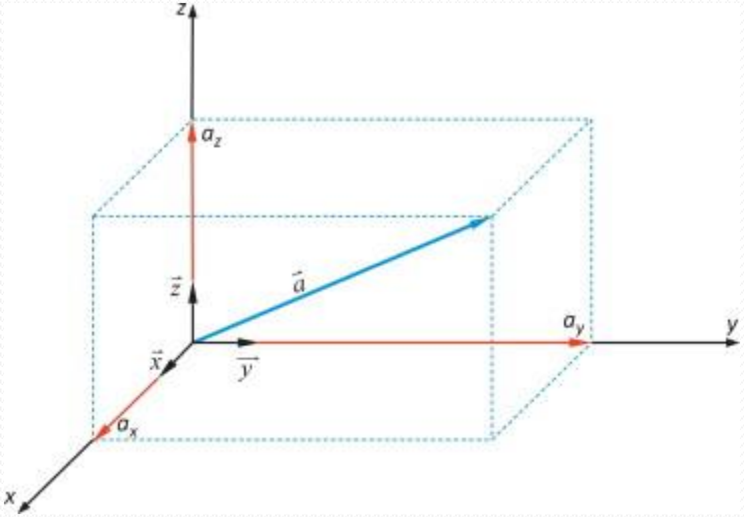


- Interrelated electric and magnetic fields traveling through space
- All electromagnetic radiation travels at $c = 3\times10^8$ m/s in vacuum – *the* cosmic speed limit!
  - real number is 299792458.0 m/s *exactly*

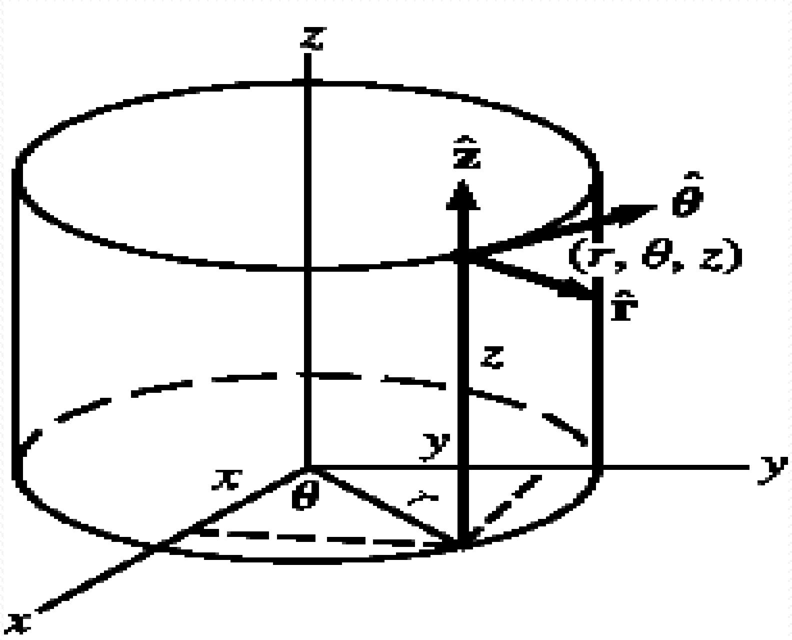
Examples of Electromagnetic Radiation

- AM and FM radio waves (including TV signals)
- Cell phone communication links
- Microwaves
- Infrared radiation
- Light
- X-rays
- Gamma rays
- What distinguishes these from one another?

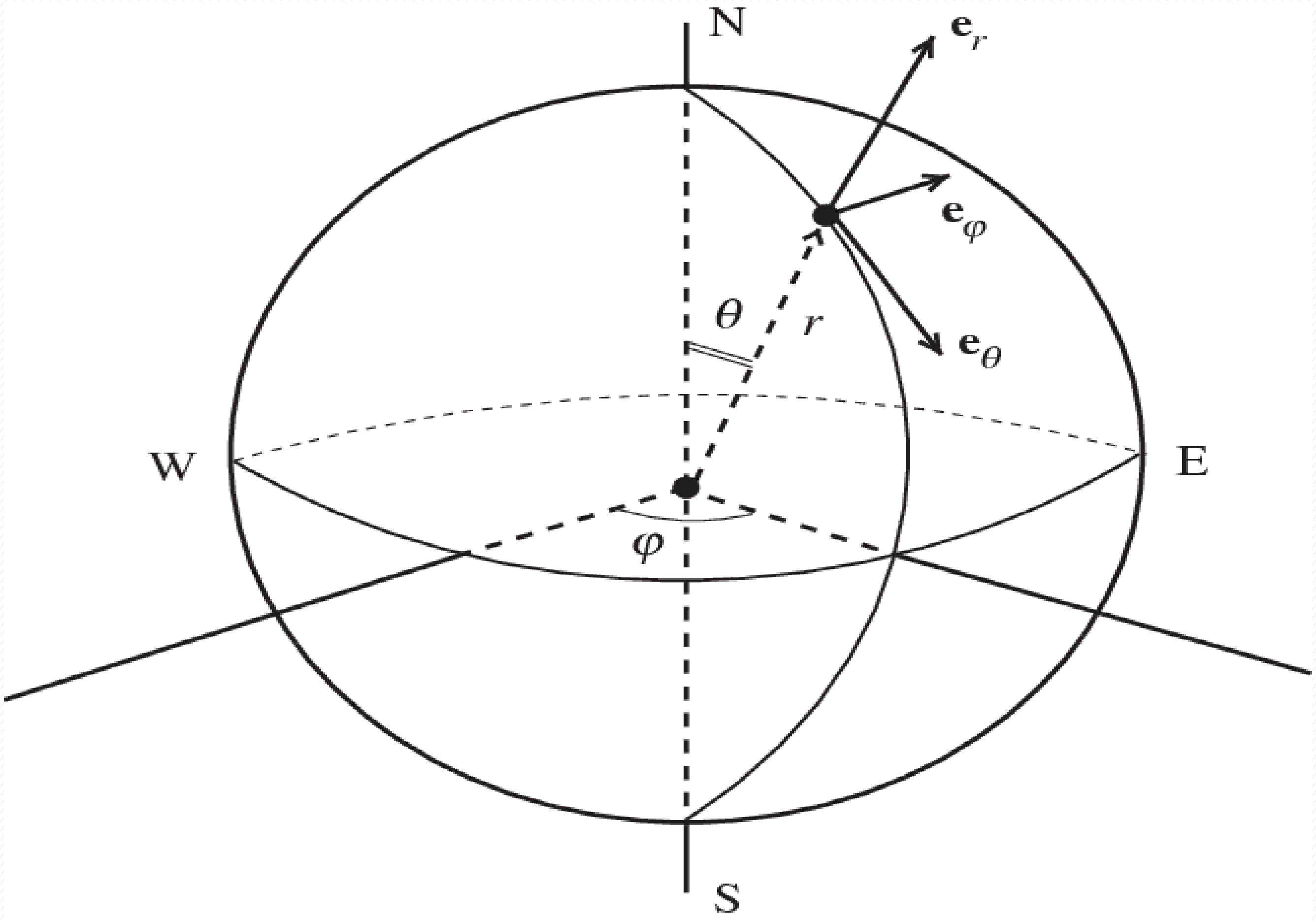
Rectangular Coordinate System



Cylindrical Coordinate System



Spherical Coordinate System



$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z \qquad \text{(cartesian)}$$

$$\nabla V = \frac{\partial V}{\partial \rho}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi + \frac{\partial V}{\partial z}\mathbf{a}_z \qquad \text{(cylindrical)}$$

$$\nabla V = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi \qquad \text{(spherical)}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o} \int_L \frac{\rho_L(\mathbf{r}')dl'}{|\mathbf{r} - \mathbf{r}'|} \qquad \text{(line charge)}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o} \int_S \frac{\rho_S(\mathbf{r}')dS'}{|\mathbf{r} - \mathbf{r}'|} \qquad \text{(surface charge)}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o} \int_v \frac{\rho_v(\mathbf{r}')dv'}{|\mathbf{r} - \mathbf{r}'|} \qquad \text{(volume charge)}$$

Surface & Volume Integral

	Surface integral	Volume integral
Diagram		
Maths description	$\Psi = \int_S \mathbf{A} \cdot d\mathbf{S}$	$\int_V \rho_v \, dv$
Result	A measure of the total flux from **vector** field passing through a given surface	A measure of the total effect of a scalar **function** i.e. temperature, inside a given volume
Information required	1. Vector field expression ***A*** 2. Surface expression	1. Scalar Function ρ_v 2. Volume expression

Integral limits depends on surface

Integral limits depends on volume

1). Vector Operators and Analysis

- Div, Grad, Curl (and all that)

- Del or nabla operator
  - In Cartesian coordinates

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

- Combining vectors in 3 ways
  - Scalar (inner) product **a**.**b** = **c** (scalar)
  - Cross (vector) product **axb** = **c** (vector)
  - Outer product (dyad) **ab** = **c** (tensor)

Scalar Product - Divergence

- **r** is a Cartesian position vector **r**=(**x**,**y**,**z**)

$$\mathbf{A}(\mathbf{r}) = \left(A_x, A_y, A_z\right)$$

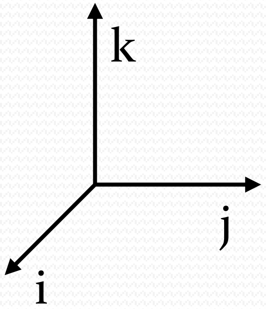
- **A** is vector function of position **r**
- Div **A** =

$$\nabla.\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

- Scalar product of **del** with **A**
- Scalar function of position

Cross Product - Curl

- Curl **A** = $\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$



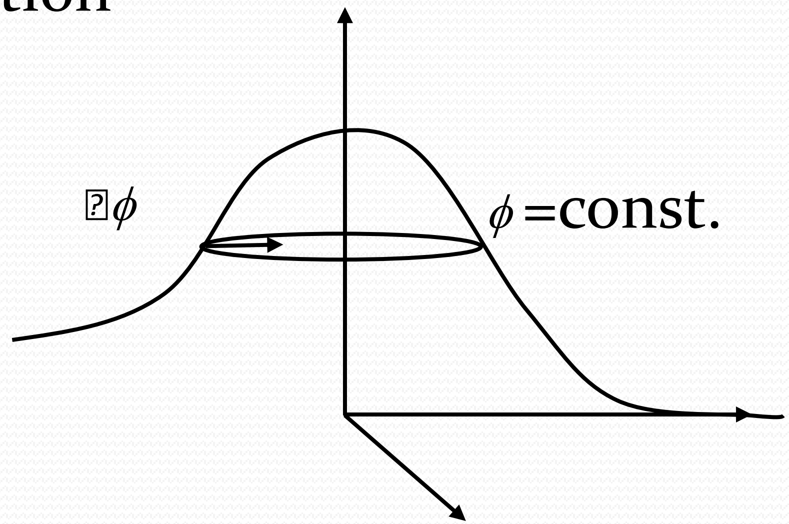
$$\nabla \times \mathbf{A} = \mathbf{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \mathbf{j}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \mathbf{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

- Cross product of **del** with **A**

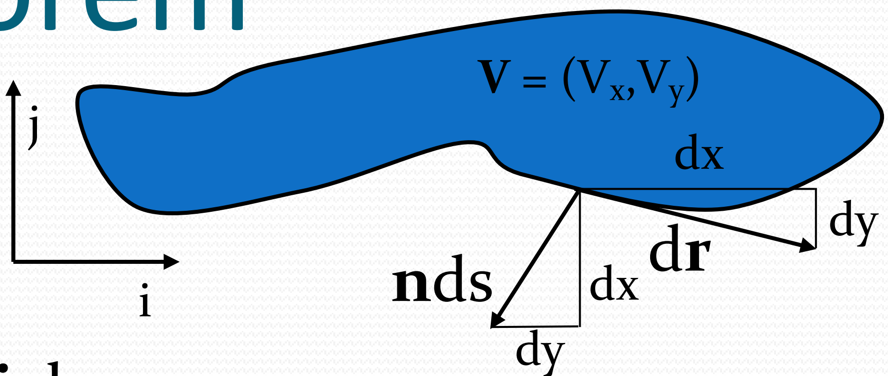
- Vector function of position

Gradient

- ϕ(x,y,z) is a scalar function of position
- Grad ϕ = ∇ϕ = $\left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z}\right)$
- Operation of del on scalar function
- Vector function of position



Divergence Theorem



- Tangent d**r** = **i** dx + **j** dy
- Outward normal **n** ds = **i** dy – **j** dx
- **n** unit vector along outward normal
- $ds = (dx^2+dy^2)^{1/2}$
- $P(x,y) = -V_y$ $Q(x,y) = V_x$

  Cartesian components of the same vector field **V**
- $P dx + Q dy = -V_y dx + V_x dy$
- $(\mathbf{i}\, V_x + \mathbf{j}\, V_y).(\mathbf{i}\, dy - \mathbf{j}\, dx) = -V_y\, dx + V_x\, dy = \mathbf{V}.\mathbf{n}\, ds$

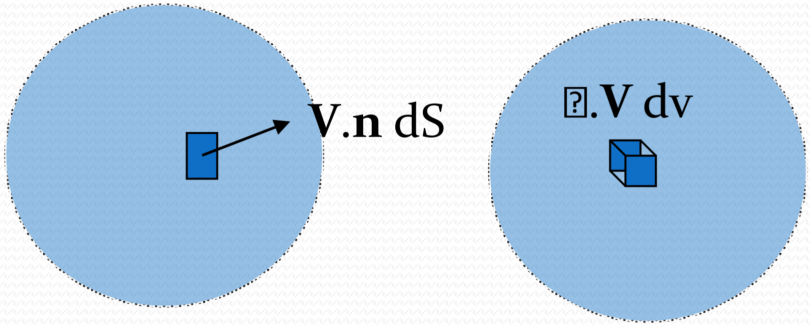
Divergence Theorem 2-D 3-D

- Apply Green's Theorem

$$\oint_C P(x,y)dx + Q(x,y)dy = \iint_A \left(\frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y} \right) dxdy$$

$$\oint_C \mathbf{V.n}\, ds = \iint_A \left(\frac{\partial V_x}{\partial x} - \frac{\partial V_y}{\partial y} \right) dxdy = \iint_A \nabla.\mathbf{V}\, dxdy$$

- *In words* - Integral of **V**.**n** ds over surface contour equals integral of div **V** over surface area



- In 3-D $\oint_S \mathbf{V.n}\, dS = \int_V \nabla.\mathbf{V}\, dv$
- Integral of **V**.**n** dS over bounding surface S equals integral of div **V** dv within volume enclosed by surface S

Curl and Stokes' Theorem

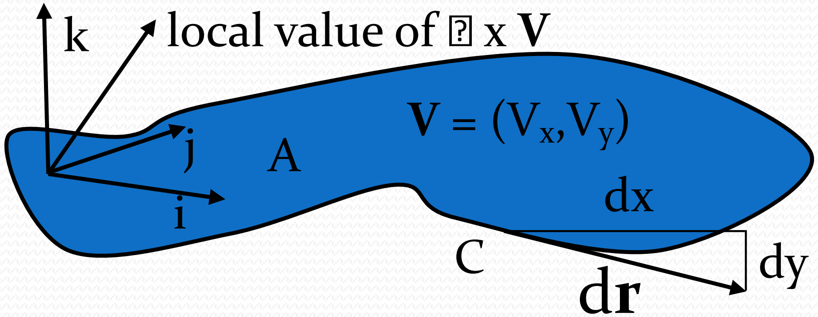
- For divergence theorem $P(x,y) = -V_y$ $\quad Q(x,y) = V_x$
- Instead choose $\quad P(x,y) = V_x \quad Q(x,y) = V_y$
- $P dx + Q dy = V_x\, dx + V_y\, dy$
- $\mathbf{V} = \mathbf{i}\, V_x + \mathbf{j}\, V_y + 0\, \mathbf{k}$

$$P(x,y)dx + Q(x,y) = V_x\, dx + V_y\, dy$$

$$P(x,y)dx + Q(x,y)dy = (\mathbf{i}\, V_x + \mathbf{j}\, V_y).(\mathbf{i}\, dx + \mathbf{j}\, dy) = \mathbf{V}\, .\, d\mathbf{r}$$

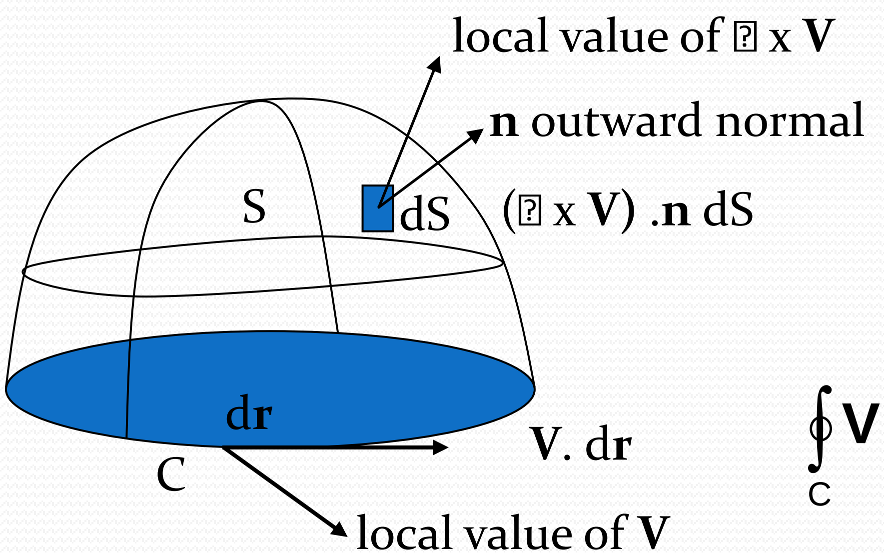
$$\frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y} = \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} = (\nabla \times \mathbf{V}).\mathbf{k}$$

$$\oint_C \mathbf{V}.d\mathbf{r} = \iint_A (\nabla \times \mathbf{V}).\mathbf{k}\, dxdy$$



Stokes' Theorem 3-D

- *In words* - Integral of (∇ x **V**) .**n** dS over surface S equals integral of **V.**d**r** over bounding contour C
- It doesn't matter which surface (blue or hatched). Direction of d**r** determined by right hand rule.



$$\oint_C \mathbf{V}.d\mathbf{r} = \iint_S (\nabla \times \mathbf{V}).\mathbf{n}\, dS$$

Stroke’s Theorem

$$\oint \mathbf{E} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = 0$$

$$\boxed{\nabla \times \mathbf{E} = 0}$$

Unit II
ELECTROSTATICS

Introduction

Electrostatics can be defined as the study of electric charges at rest. Electric fields have their sources in electric charges.

(Note: Almost all real electric fields vary to some extent with time. However, for many problems, the field variation is slow and the field may be considered as static. For some other cases spatial distribution is nearly same as for the static case even though the actual field may vary with time. Such cases are termed as quasi-static.)

In this chapter we first study two fundamental laws governing the electrostatic fields, viz, (1) Coulomb's Law and (2) Gauss's Law. Both these law have experimental basis. Coulomb's law is applicable in finding electric field due to any charge distribution, Gauss's law is easier to use when the distribution is symmetrical.

Coulomb's Law

- Coulomb determined
  - Force is attractive if charges are opposite sign
  - Force proportional to the product of the charges q_1 and q_2 along the lines joing them
  - Force inversly proportional square of the distance
- I.e.
  - $|F_{12}| \propto |Q_1|\ |Q_2| / r_{12_2}$
  - or
  - $|F_{12}| = k\ |Q_1|\ |Q_2| / r_{12_2}$

Coulomb's Law

- Units of constant can be determined from Coulomb's Law
- Colomb (and others since) have determined this constant which (in a vacuum) in SI units is
  - k = 8.987.5x10^{9} Nm^{2}C^{-2}
- k is normally expressed as $k = 1/4\pi\varepsilon_0$
  - where is the permittivity of free space

Coulomb's Law

The equation for the magnitude of the Coulomb force between two point charges Q_1 and Q_2 in a vacuum is given by

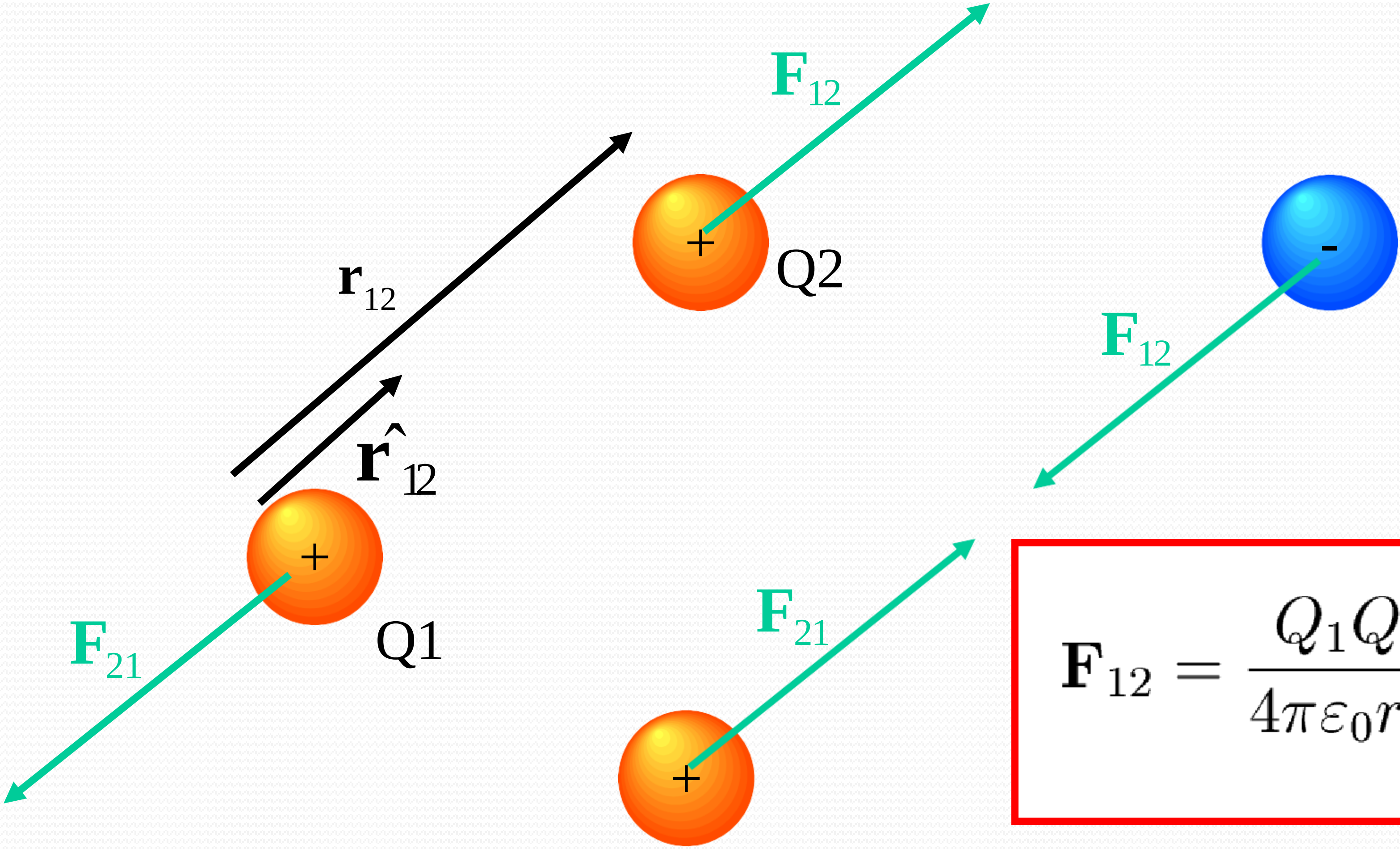
$$|\mathbf{F}_{12}| = \frac{|Q_1 Q_2|}{4\pi\varepsilon_0 r_{12}^2}$$

where

$\lvert Q_1\rvert$	is	the magnitude of the charge Q_1 in coulombs (C)
$\lvert Q_2\rvert$	is	the magnitude of the charge Q_2 in coulombs (C)
$\mathbf{F}_{12}$	is	the electrical force acting on the charge Q_1 due to charge Q_2 in newtons (N)
r_{12}	is	the distance between the point charges Q_1 and Q_2 in metres (m)
ε_0	is	the permittivity of free space in $C^2\ N^{-1}\ m^{-2}$
$\frac{1}{4\pi\varepsilon_0}$	is	the Coulomb constant in $N\ m^2\ C^{-2}$.

The direction of the force $\mathbf{F}_{12}$ is determined by the sign of the charges; the force is attractive if the charges have opposite signs, and repulsive if the charges have the same sign.

Vector form of Coulomb's Law



$$\mathbf{F}_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 r_{12}^2}\hat{\mathbf{r}}_{12},$$

Coulomb's Law vs Newton's Law of Gravity

$$\mathbf{F}_{12} = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{|\mathbf{r}_{12}|^2}\hat{\mathbf{r}}_{12}$$

- Attractive or repulsive
- *1/r*²
- very strong
- only relevant relatively local scales

$$\mathbf{F}_{12} = -G\frac{m_1 m_2}{|\mathbf{r}_{12}|^2}\hat{\mathbf{r}}_{12}$$

- Always attractive
- *1/r*²
- very weak $\frac{e^2}{4\pi\varepsilon_0} >> -Gm^2$
- important on very large scales, planets, the Universe

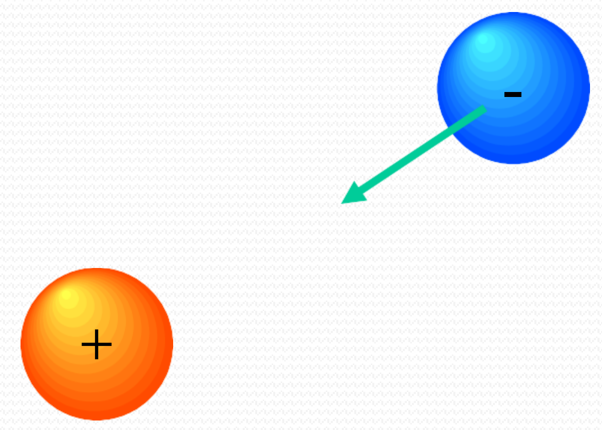
Two spheres

Electric Field

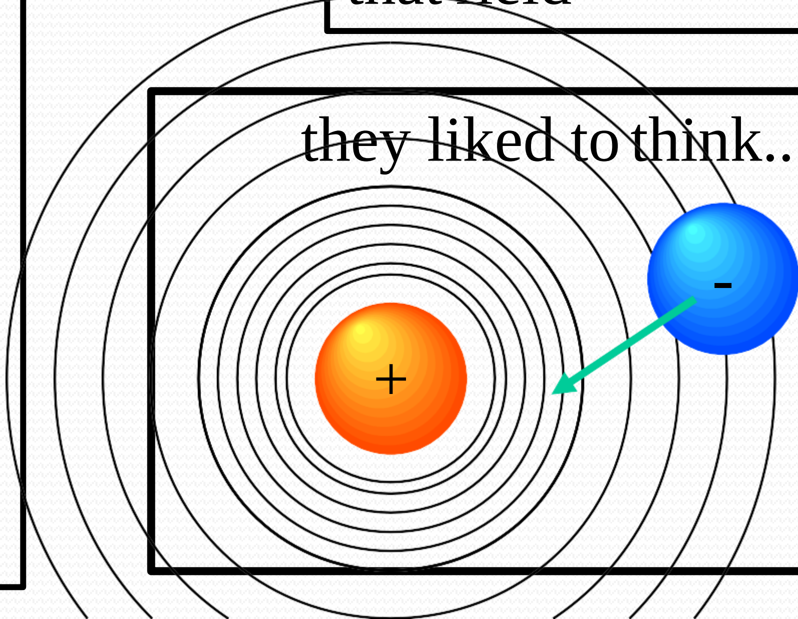
Physicists did not like the concept of "action at a distance" i.e. a force that was "caused" by an object a long distance away

They preferred to think of an object producing a "field" and other objects interacting with that field

Thus rather than ...



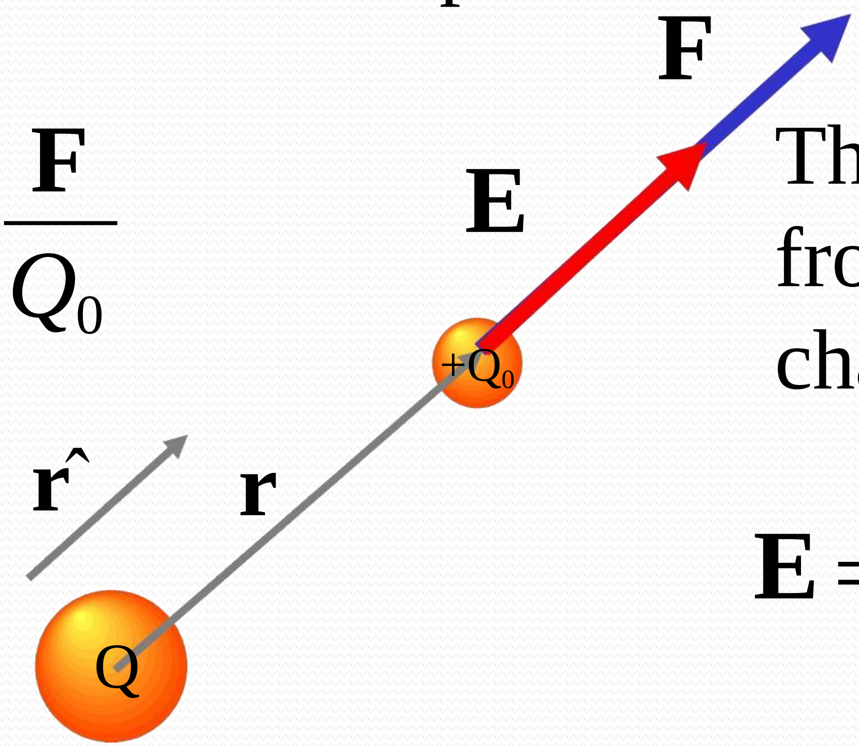
they liked to think...



Electric Field

Electric Field E is defined as the force acting on a test particle divided by the charge of that test particle

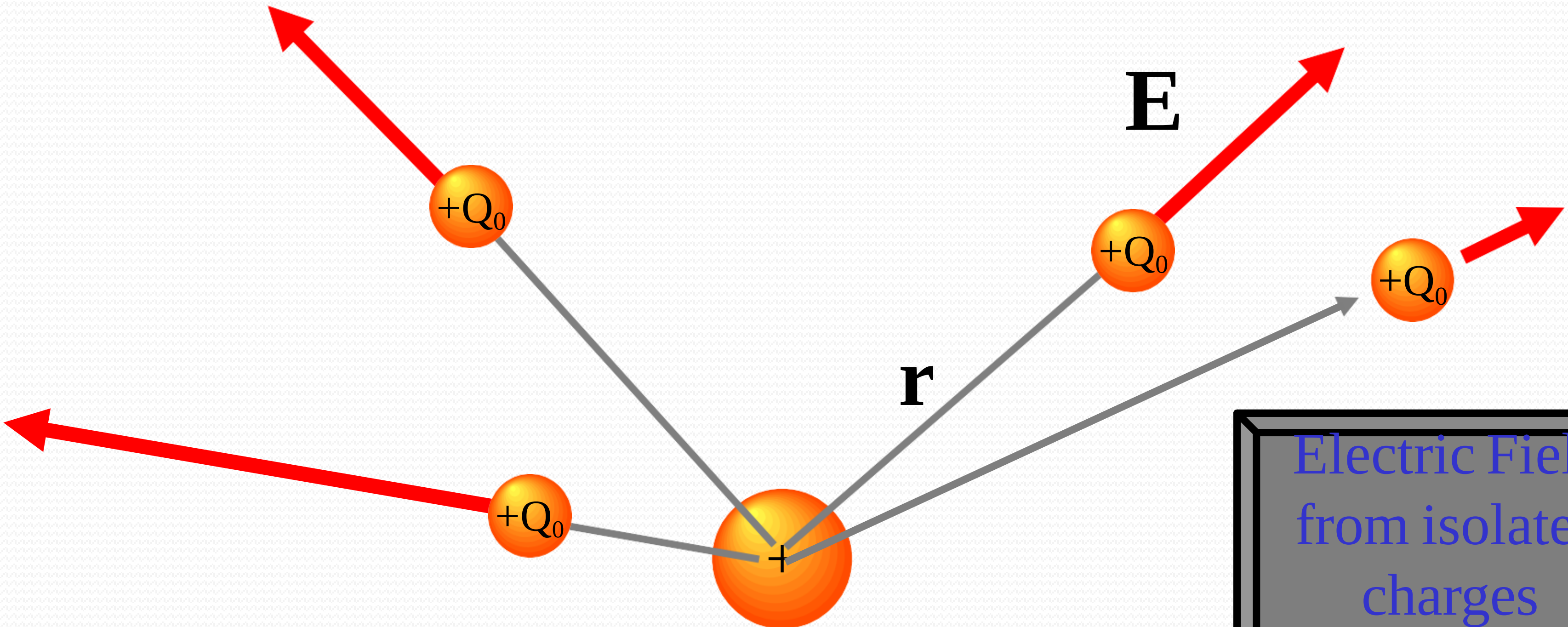
$$\mathbf{E} = \frac{\mathbf{F}}{Q_0}$$



Thus Electric Field from a single charge is

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{|\mathbf{r}|^2}\hat{\mathbf{r}}$$

Electric Field of a single charge



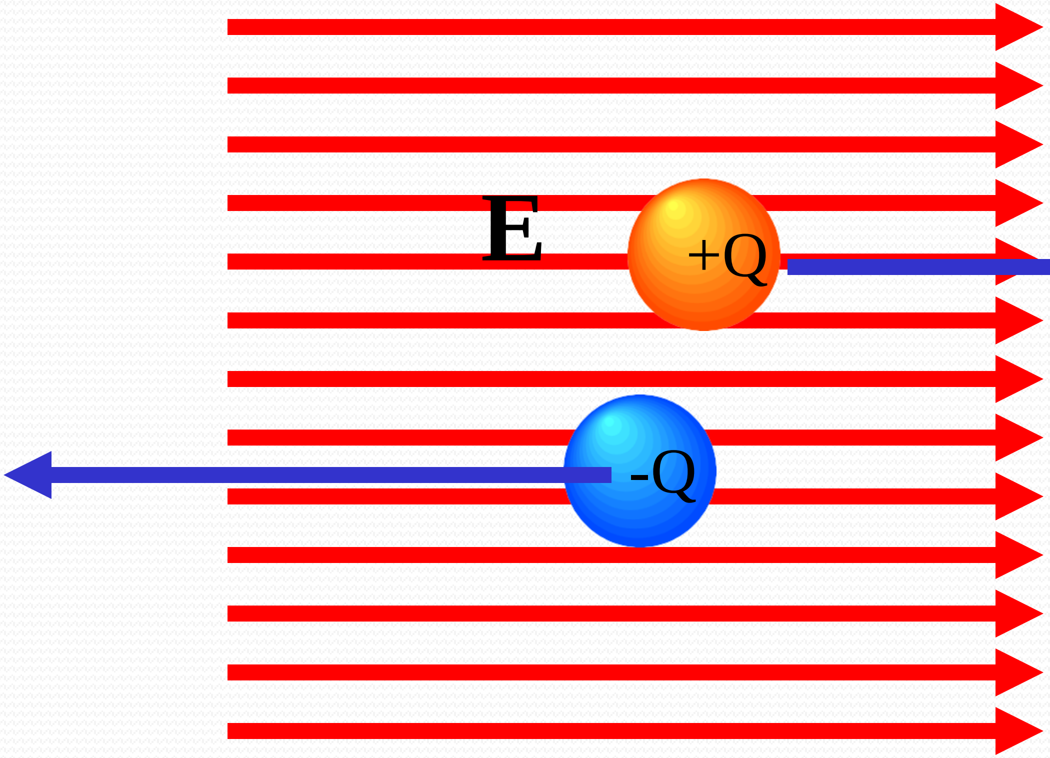
Electric Field from isolated charges (interactive)

Electric field from test particles

Note: the Electric Field is defined everywhere, even if there is no test charge is not there.

Charged particles in electric field

Using the Field to determine the force



$$\mathbf{F} = Q\mathbf{E}$$

$$\mathbf{F} = Q\mathbf{E}$$

Electric Field as a vector field

The Electric Field is one example of a Vector Field

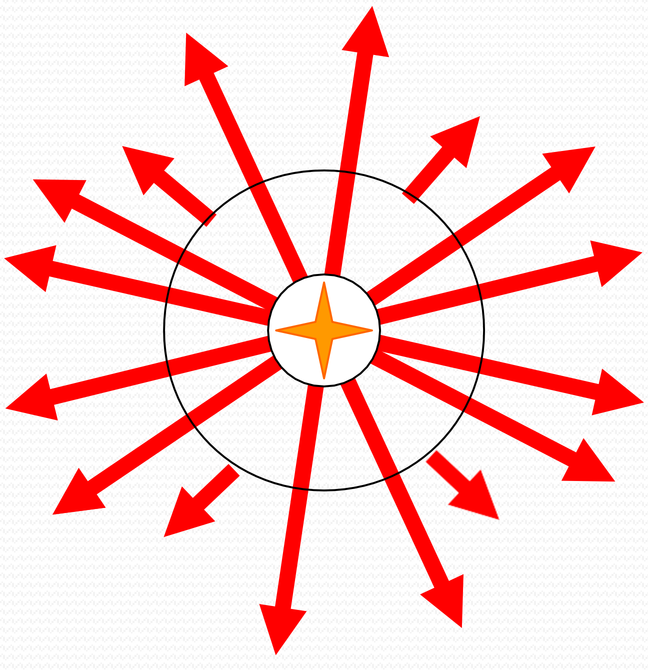
A “field” (vector or scalar) is defined everywhere

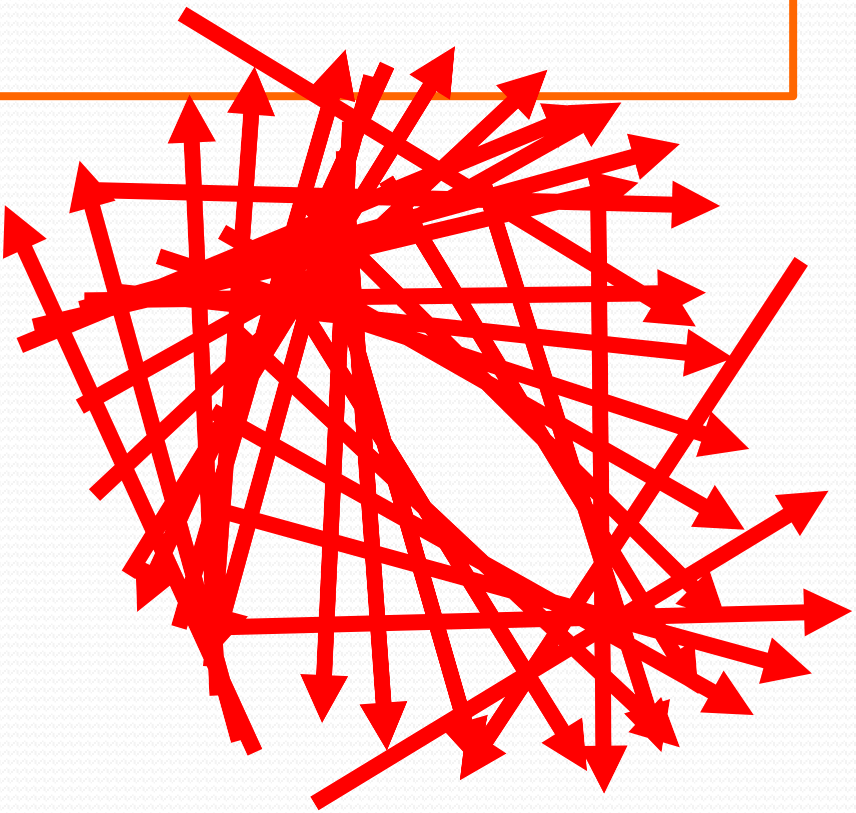
A vector field has direction as well as size

The Electric Field has units of N/C

Representation of the Electric Field

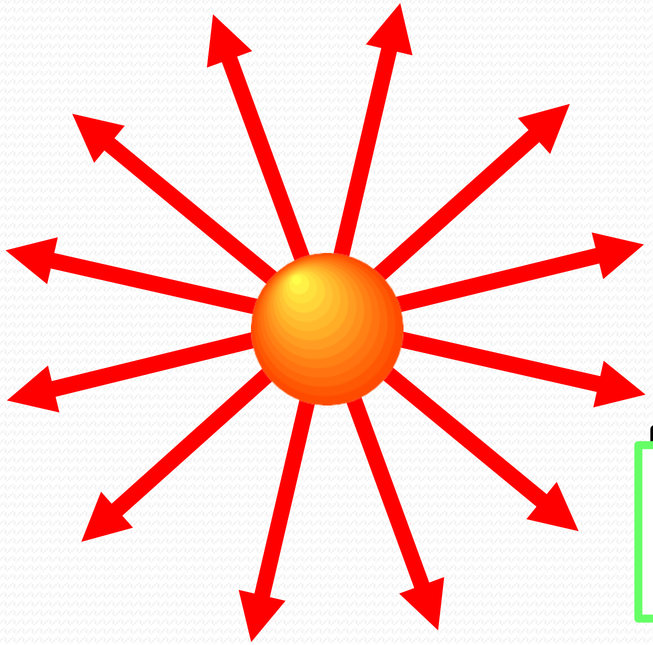
It would be difficult to represent the electric field by drawing vectors whose direction was the direction of the field and whose length was the size of the field everywhere





Representation of the Electric Field

Instead we choose to represent the electric field with lines whose direction indicates the direction of the field



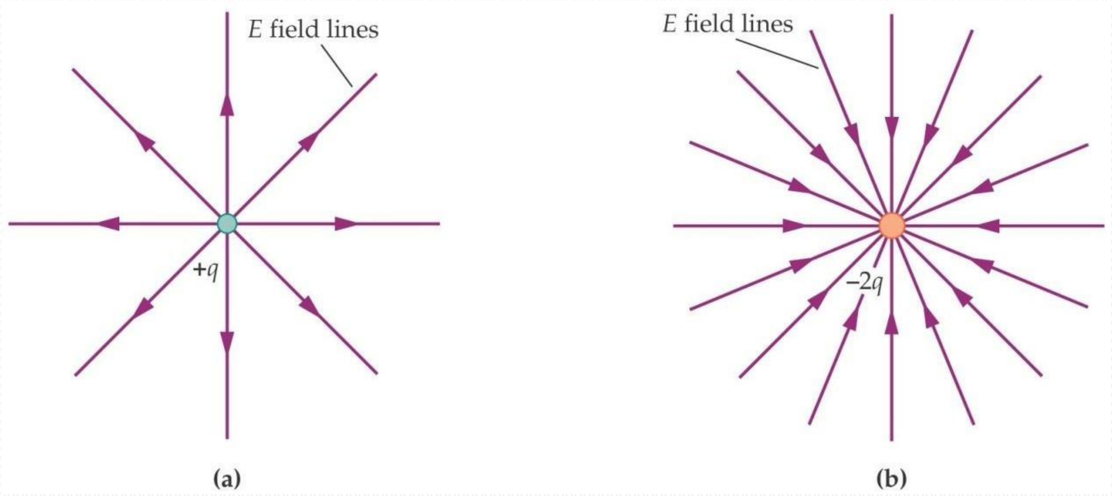
Notice that as we move away from the charge, the density of lines decreases

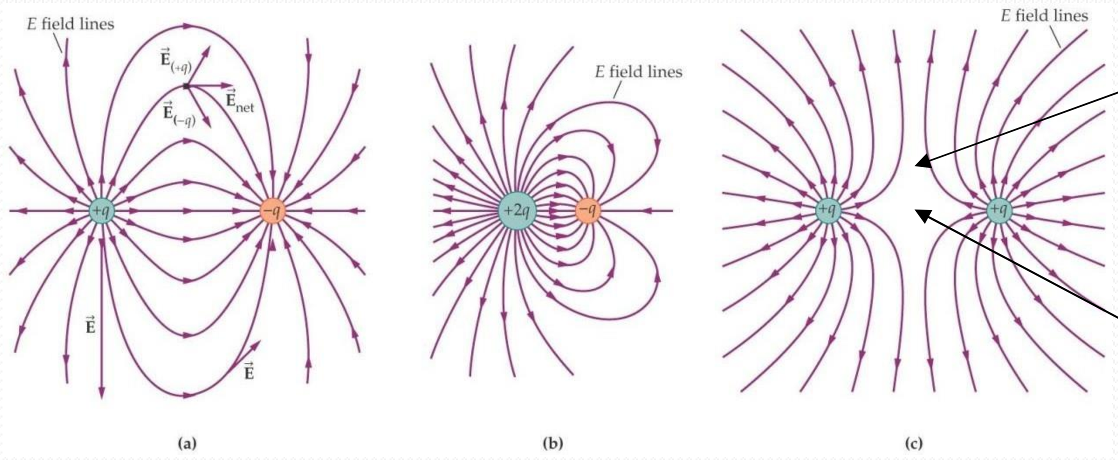
These are called Electric Field Lines

Drawing Electric Field Lines

- The lines must begin on positive charges (or infinity)
- The lines must end on negative charges (or infinity)
- The number of lines leaving a +ve charge (or approaching a -ve charge) is proportional to the magnitude of the charge
- electric field lines cannot cross

Electric Field Lines

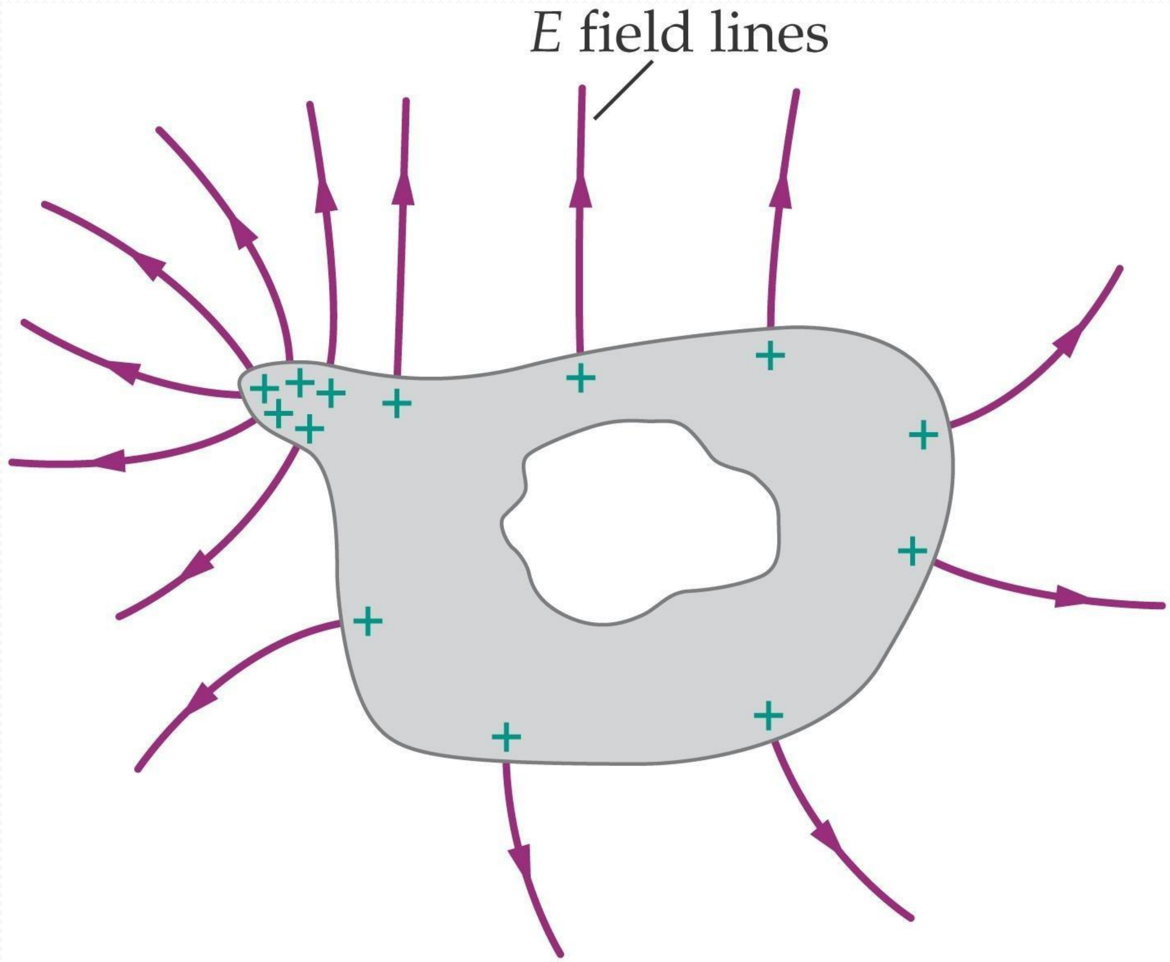




Field is *not* zero here

Field is zero at midpoint

Field lines for a conductor

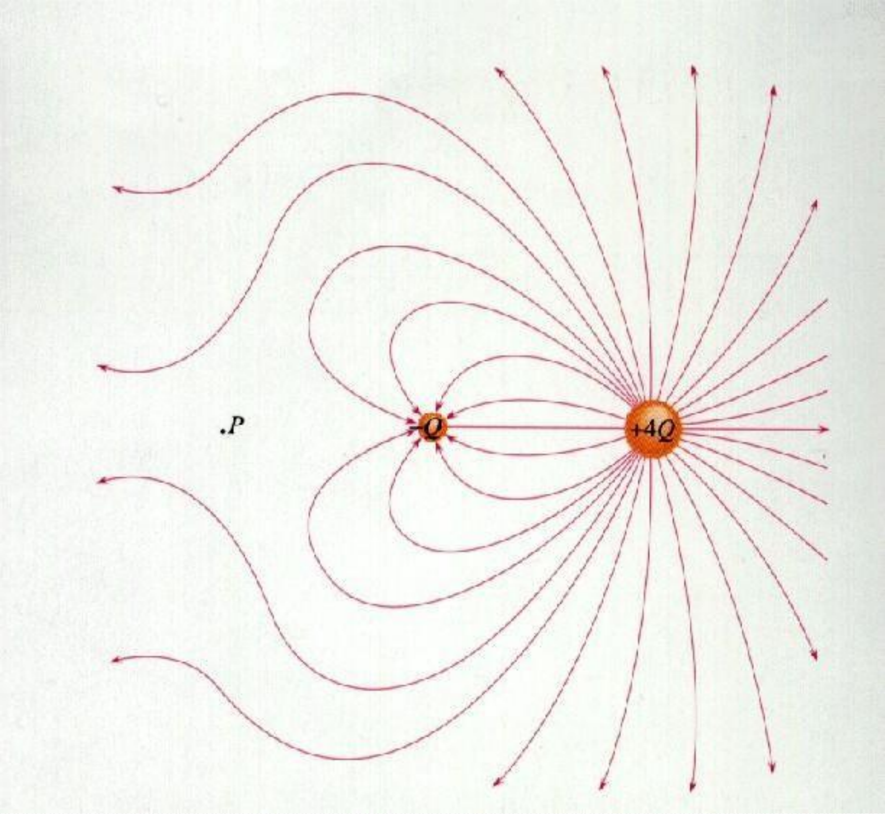


Drawing Electric Field Lines: Examples

From Electric field vectors to field lines

Field lines from all angles

Field lines representation



Electric Field Lines

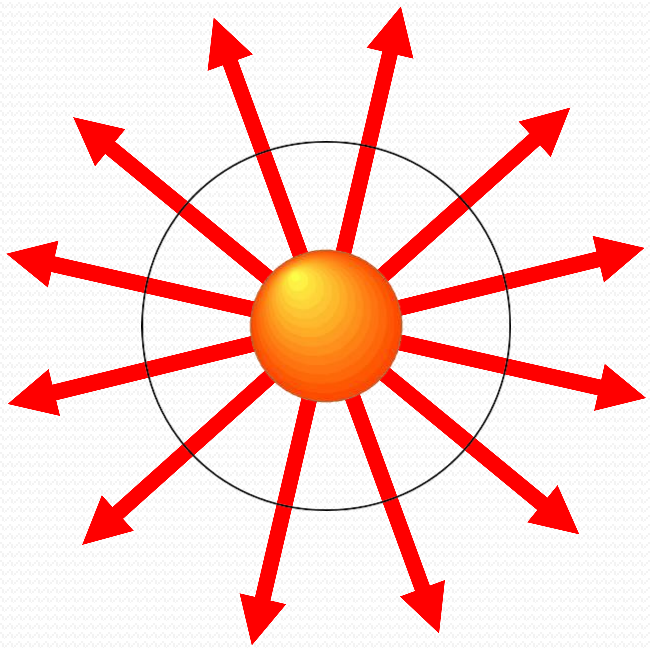
Define $\rho \equiv \frac{N_{\text{lines}}}{A}$ $\Rightarrow$ $\rho = \frac{\square N}{4\pi r^2}$

since $N_{\text{lines}} \propto Q$

$\rho \propto \frac{Q}{4\pi r^2}$

we know

$$|\mathbf{E}| = \frac{1}{4\pi\varepsilon_0} \frac{Q}{|\mathbf{r}|^2}$$

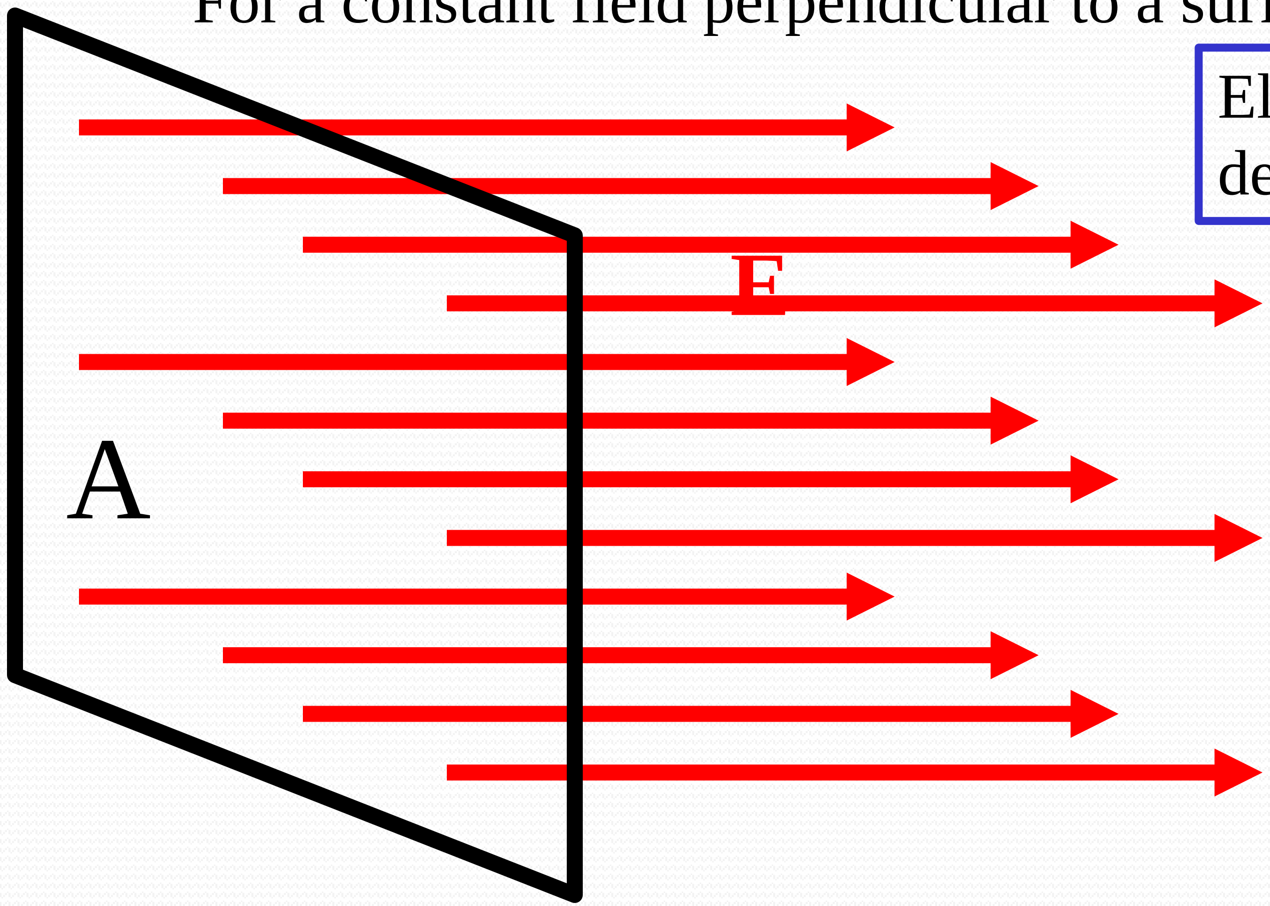


The number density of field lines is

$$|\mathbf{E}| \propto \rho$$

Electric Flux: Field Perpendicular

For a constant field perpendicular to a surface A

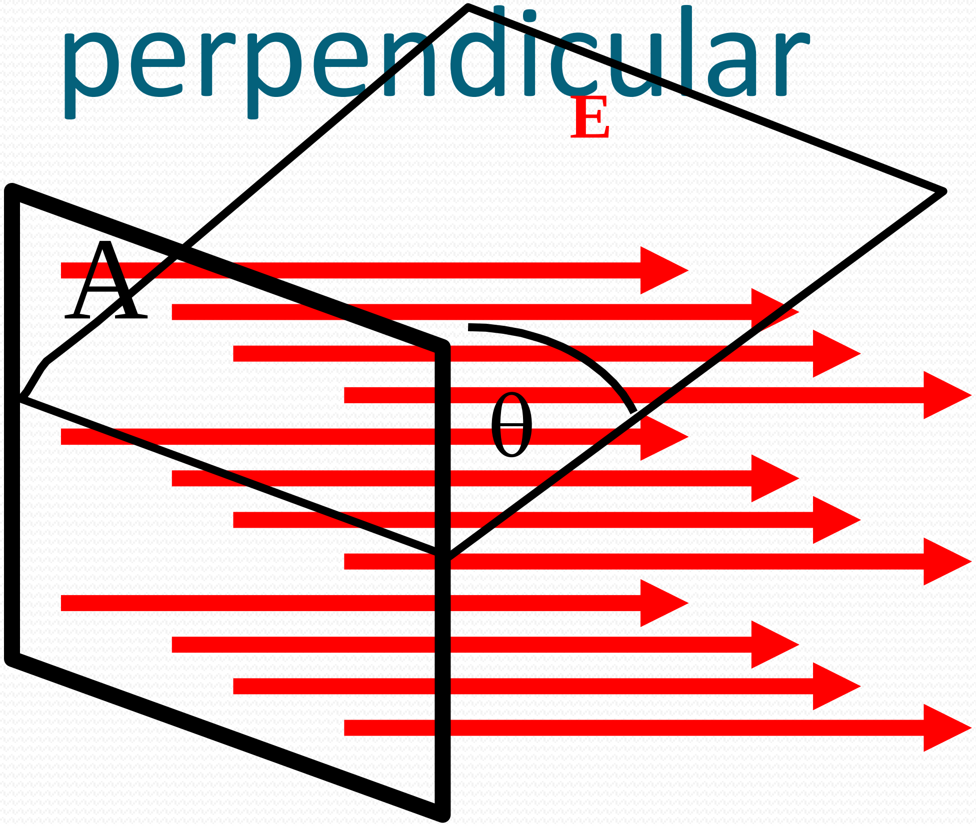


Electric Flux is defined as

$$\Phi = |\mathbf{E}|A$$

Electric Flux: Non perpendicular

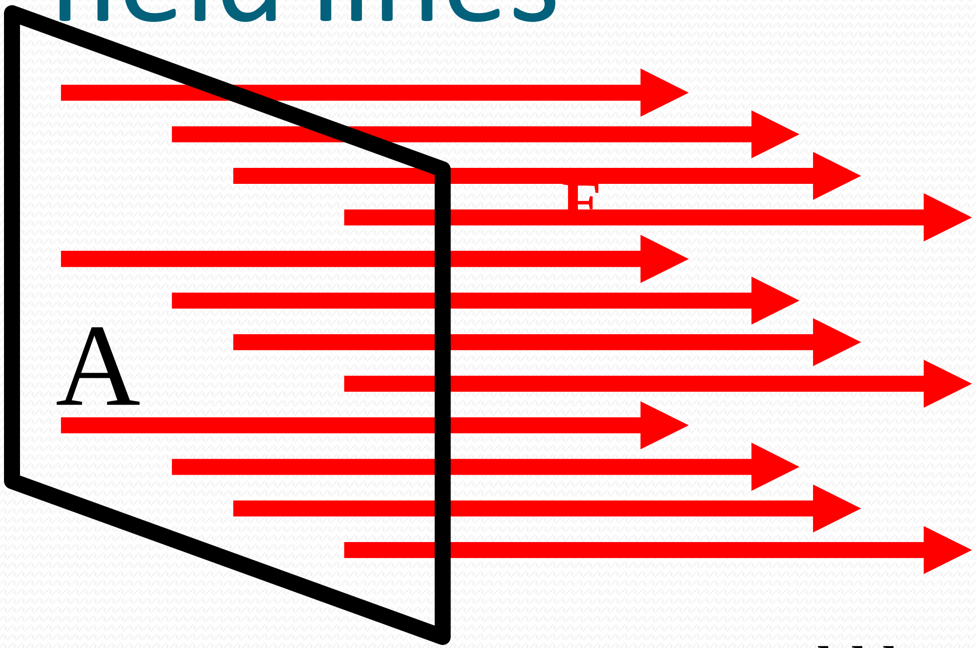
For a constant field NOT perpendicular to a surface A



Electric Flux is defined as

$$\Phi = |\mathbf{E}| A \cos\theta$$

Electric Flux: Relation to field lines



$$\Phi = |\mathbf{E}|A$$

Field line density $\rho \propto |\mathbf{E}|$

Field line density × Area $\rho A \propto |\mathbf{E}|A$

FLUX

Number of flux lines $N \propto \Phi$

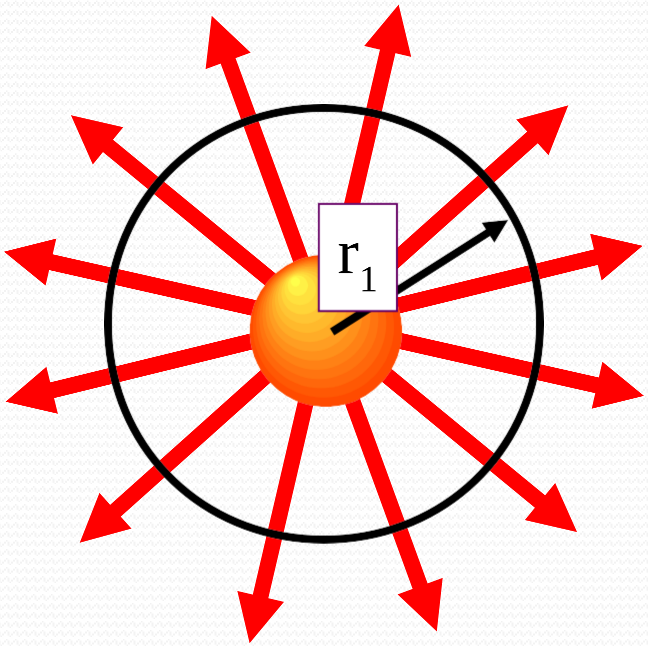
Gauss's Law

Relates flux through a closed surface
to
charge within that surface

Flux through a sphere from a point charge

The electric field around a point charge

$$|\mathbf{E}| = \frac{1}{4\pi\varepsilon_0}\frac{Q}{|\mathbf{r}_1|^2}$$



Thus the flux on a sphere is E × Area

E · Area

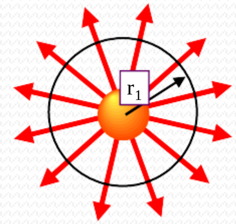
$$\Phi = \frac{1}{4\pi\varepsilon_0}\frac{Q}{|\mathbf{r}_1|^2} \times 4\pi|\mathbf{r}_1|^2$$

Cancelling we get

$$\Phi = \frac{Q}{\varepsilon_0}$$

Now we change the radius of sphere

Flux through a sphere from a point charge

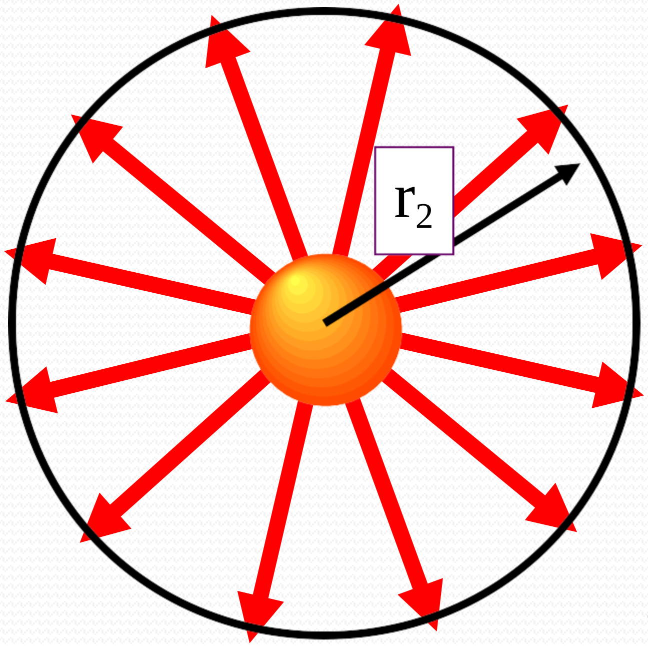


The electric field around a point charge $|\mathbf{E}| = \frac{1}{4\pi\varepsilon_0} \frac{Q}{|\mathbf{r}_1|^2}$

Thus the flux on a sphere is E × Area $\Phi = \frac{1}{4\pi\varepsilon_0} \frac{Q}{|\mathbf{r}_1|^2} \times 4\pi |\mathbf{r}_1|^2$

E Area

Cancelling we get $\Phi = \frac{Q}{\varepsilon_0}$



$$|\mathbf{E}| = \frac{1}{4\pi\varepsilon_0} \frac{Q}{|\mathbf{r}_2|^2}$$

$$\Phi_2 = \frac{1}{4\pi\varepsilon_0} \frac{Q}{|\mathbf{r}_2|^2} \times 4\pi |\mathbf{r}_2|^2$$

$$\Phi_2 = \frac{Q}{\varepsilon_0}$$

The flux is the same as before

$$\Phi_2 = \Phi_1 = \frac{Q}{\varepsilon_0}$$

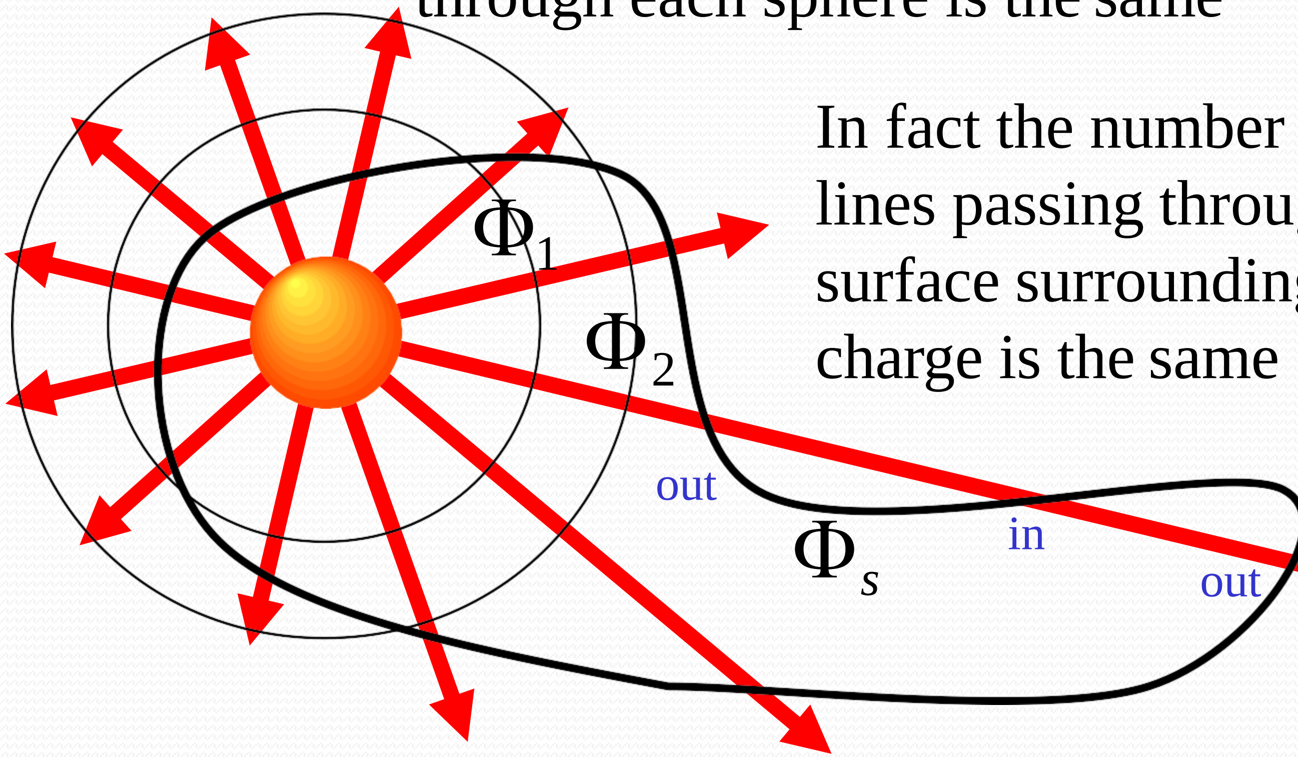
Flux lines & Flux

Just what we would expect because the number of field lines passing through each sphere is the same

$$N \propto \Phi \qquad \Phi \propto N$$

and number of lines passing through each sphere is the same

$$\Phi_S = \Phi_2 = \Phi_1 = \frac{Q}{\varepsilon_0}$$



In fact the number of flux lines passing through any surface surrounding this charge is the same

even when a line passes in and out of the surface it crosses out once more than in

What is Gauss's Law?

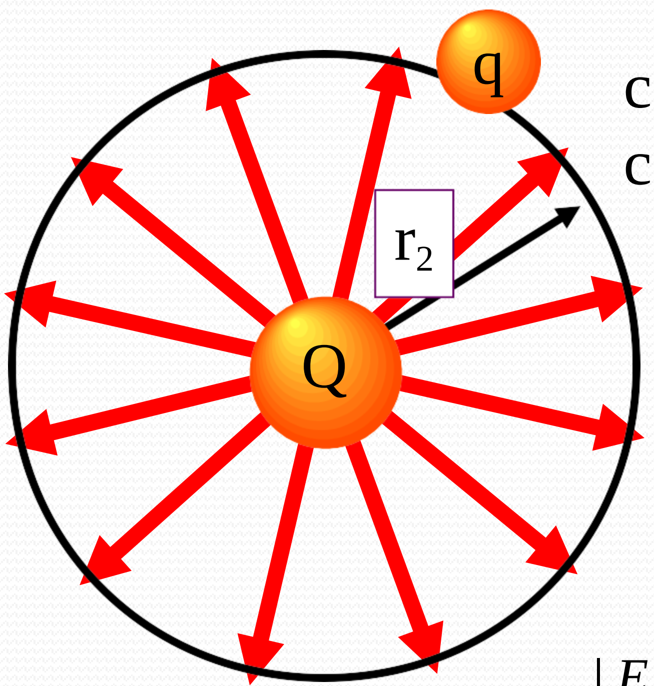
Gauss's Law does not tell us anything new, it is NOT a new law of physics, but another way of expressing Coulomb's Law

Gauss's Law is sometimes easier to use than Coulomb's Law, especially if there is lots of symmetry in the problem

Example of using Gauss's Law 1

oh no! I've just forgotten Coulomb's Law!

Not to worry I remember Gauss's Law



consider spherical surface centred on charge

$$\Phi = \frac{Q}{\varepsilon_0}$$

By symmetry **E** is ⊥ to surface

$$\Phi = |E|A = \frac{Q}{\varepsilon_0} \qquad = |E|\,4\pi r^2 = \frac{Q}{\varepsilon_0}$$

$$|E| = \frac{1}{4\pi r^2}\frac{Q}{\varepsilon_0} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$$

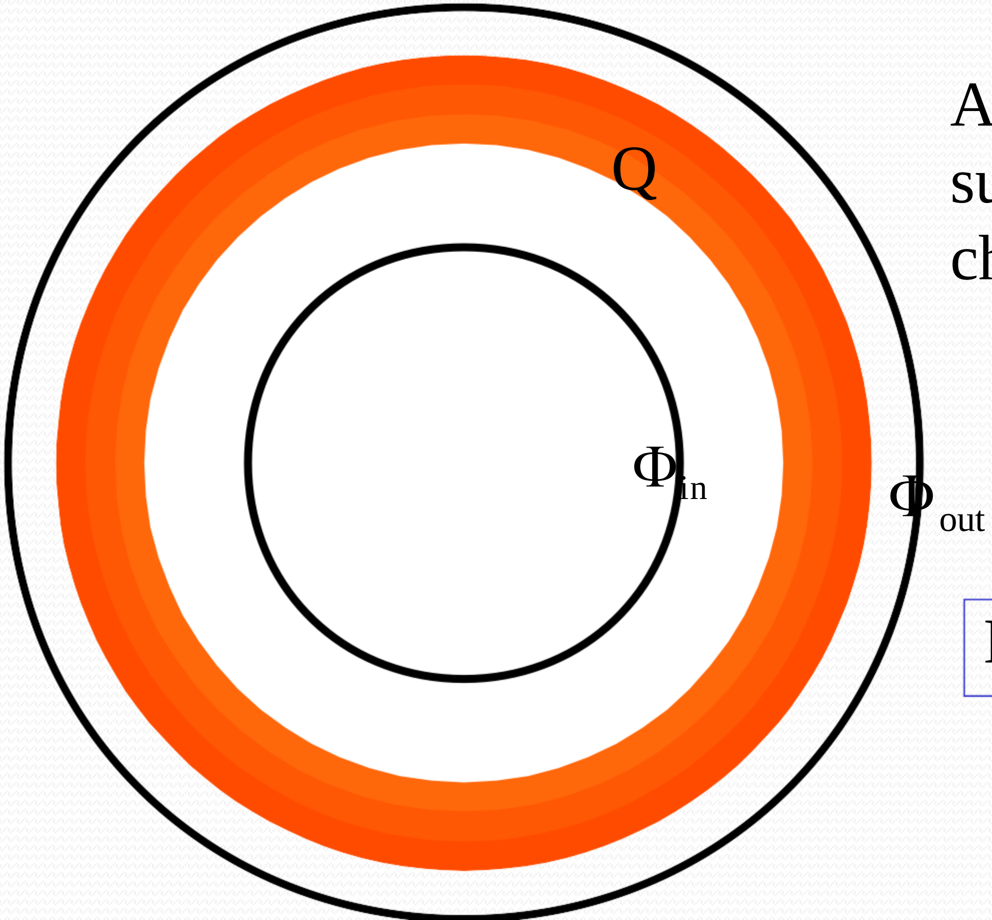
F=qE

$$F = \frac{1}{4\pi r^2}\frac{qQ}{\varepsilon_0}$$

Phew!

Example of using Gauss's Law 2

What's the field around a charged spherical shell?



Again consider spherical surface centred on charged shell

Outside

$$\Phi_{out} = \frac{Q}{\varepsilon_0}$$

So as e.g. 1 $|E| = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$

Inside

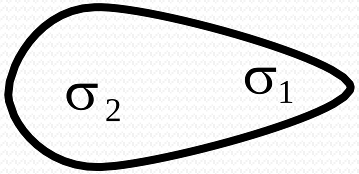
charge within surface = 0

$\Phi_{in} = 0$ $\qquad E = 0$

Properties of Conductors

For a conductor in electrostatic equilibrium

1. E is zero within the conductor
2. Any net charge, Q, is distributed on surface (surface charge density $\sigma = Q/A$)
3. E immediately outside is $\perp$ to surface
4. σ is greatest where the radius of curvature is smaller



$\sigma_1 >> \sigma_{21}$

1. E is zero within conductor

If there is a field in the conductor, then the free electrons would feel a force and be accelerated. They would then move and since there are charges moving the conductor would not be in electrostatic equilibrium

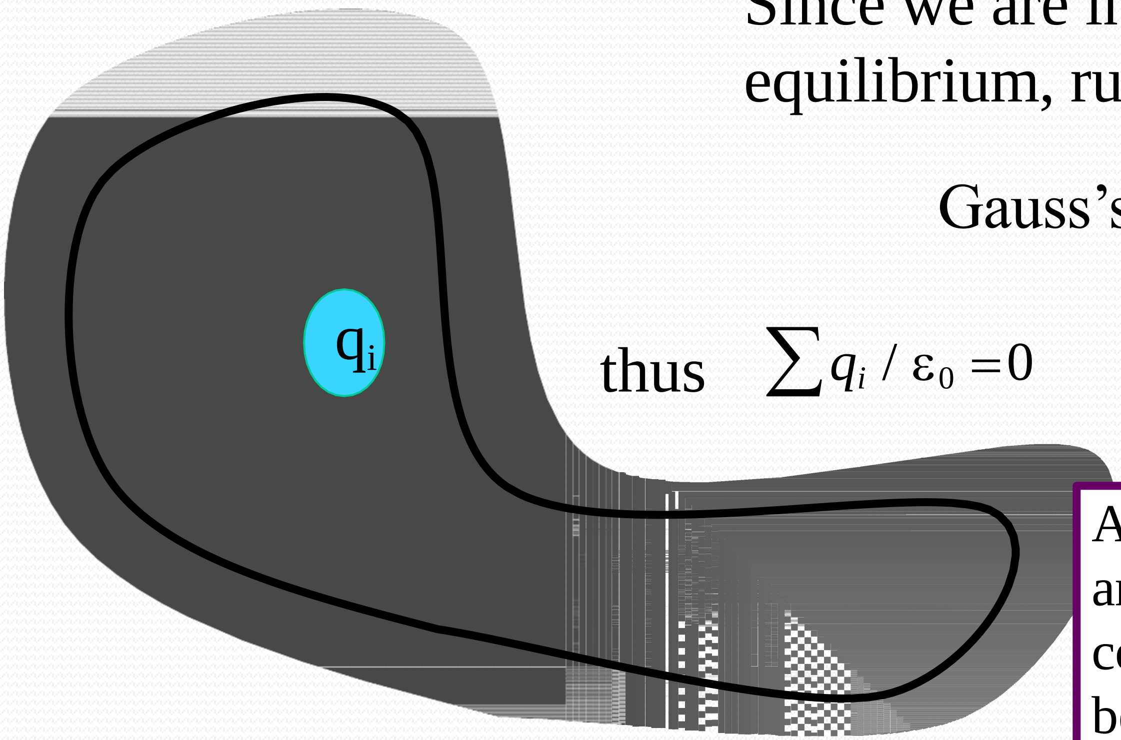
Thus E=0

2. Any net charge, Q, is distributed on surface

Consider surface S below surface of conductor

Since we are in a conductor in equilibrium, rule 1 says E=0, thus Φ=0

Gauss's Law $\Phi = EA = \sum q / \varepsilon_0$

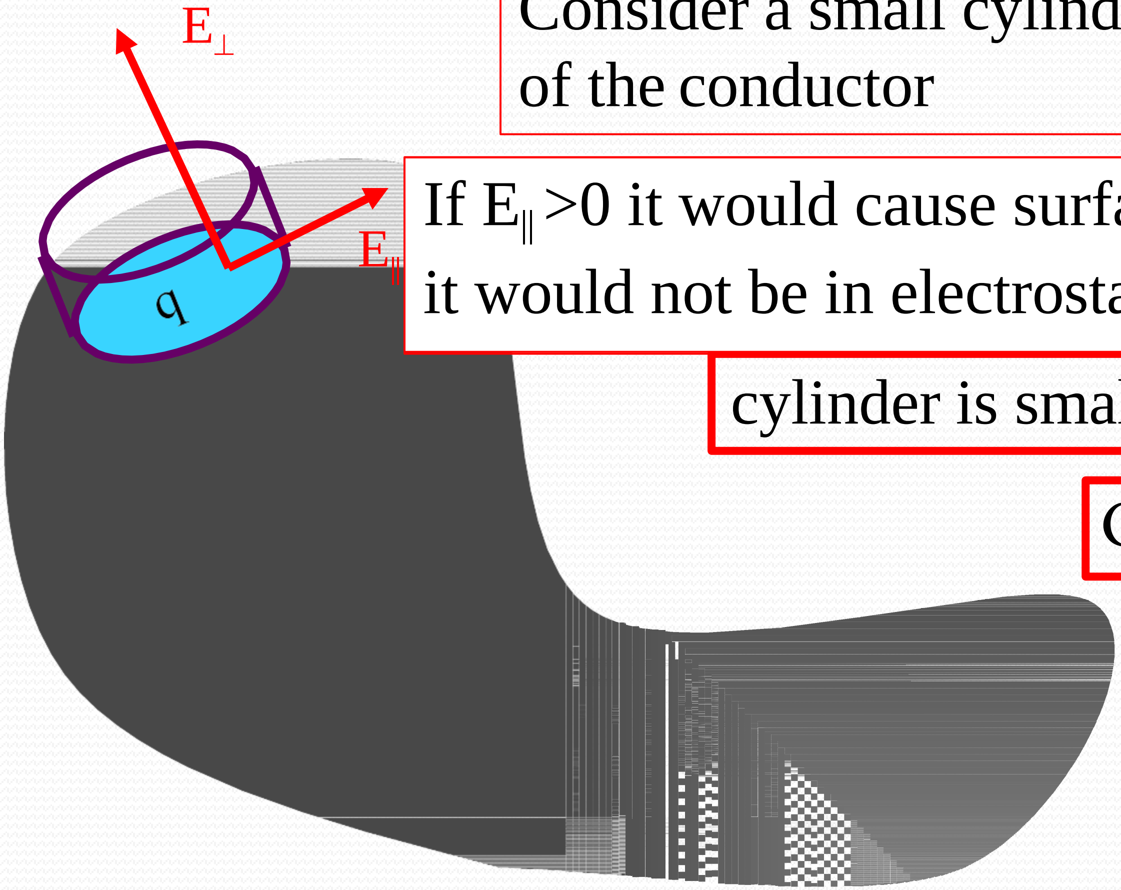


thus $\sum q_i / \varepsilon_0 = 0$

So, net charge within the surface is zero

As surface can be drawn arbitrarily close to surface of conductor, all net charge must be distributed on surface

3. E immediately outside is $\perp$ to surface



Consider a small cylindrical surface at the surface of the conductor

If $E_\parallel$ >0 it would cause surface charge q to move thus it would not be in electrostatic equilibrium, thus $E_\parallel$=0

cylinder is small enough that E is constant

Gauss's Law $\Phi = EA = q/\varepsilon$

thus $E = q/A\varepsilon$

$$E_\perp = \sigma/\varepsilon$$

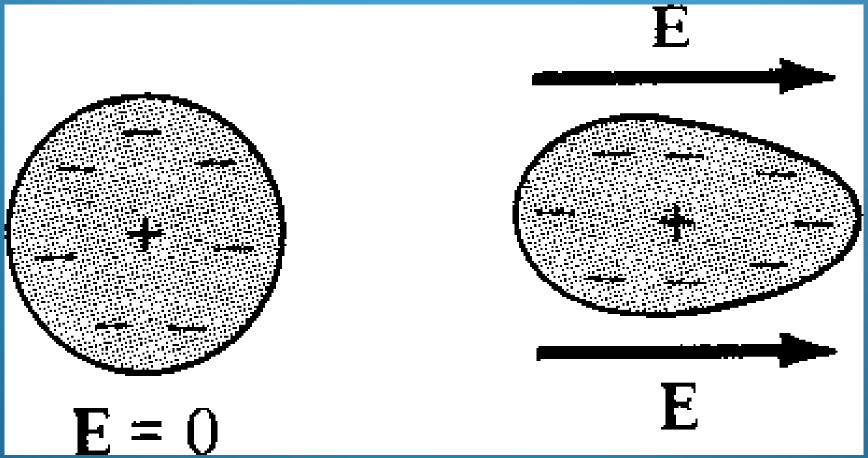
Electric Field inside a dielectric

- Material
- Dielectric- Conductor And Dielectric – Dielectric Boundary Conditions
- Capacitance
- Current Density
- Ohm's Law

Equation of Continuity

Electric Field inside Diectric medium

positive charge +Q(nucleus) as in Figure



the force $F_+ = QE$ while the negative charge is
positive charge is displaced its
displaced in the opposite direction from by the force $F\_ = -QE$

- A dipole results from the displacement of the charges and the dielectric is said to be polarized.

- $$\mathbf{p} = Q\mathbf{d}$$

$$Q_1\mathbf{d}_1 + Q_2\mathbf{d}_2 + \cdots + Q_N\mathbf{d}_N = \sum_{k=1}^{N} Q_k\mathbf{d}_k$$

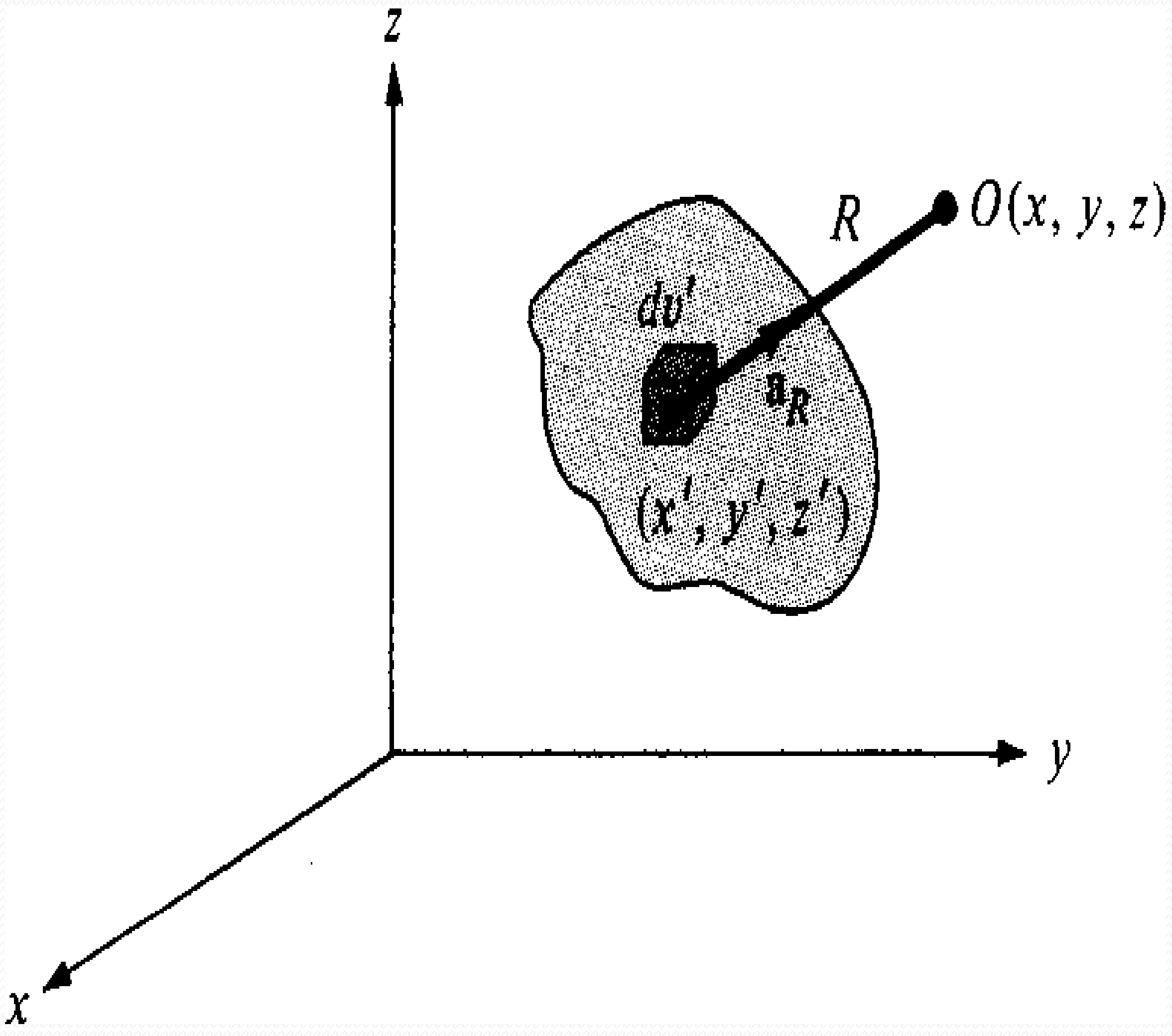
$$\mathbf{P} = \frac{\lim_{\Delta v \to 0} \sum_{k=1}^{N} Q_k\mathbf{d}_k}{\Delta v}$$

The major effect of the electric field E on a dielectric is the creation of dipole moments that align themselves in the direction of E.

$$dV = \frac{\mathbf{P} \cdot \mathbf{a}_R \, dv'}{4\pi\varepsilon_o R^2}$$

$$\nabla' = \frac{1}{R} = \frac{\mathbf{a}_R}{R^2}$$

$$\frac{\mathbf{P} \cdot \mathbf{a}_R}{R^2} = \mathbf{P} \cdot \nabla' \left(\frac{1}{R} \right)$$


z
R
O(x, y, z)
dv'
a_R
(x', y', z')
y
x

$$\nabla' \cdot f\mathbf{A} = f\nabla' \cdot \mathbf{A} + \mathbf{A} \cdot \nabla' f,$$

$$\frac{\mathbf{P} \cdot \mathbf{a}_R}{R^2} = \nabla' \cdot \frac{\mathbf{P}}{R} - \frac{\nabla' \cdot \mathbf{P}}{R}$$

$$V = \int_{v'} \frac{1}{4\pi\varepsilon_o} \left[\nabla' \cdot \frac{\mathbf{P}}{R} - \frac{1}{R} \nabla' \cdot \mathbf{P} \right] dv'$$

$$V = \int_{S'} \frac{\mathbf{P} \cdot \mathbf{a}'_n}{4\pi\varepsilon_o R} dS' + \int_{v'} \frac{-\nabla' \cdot \mathbf{P}}{4\pi\varepsilon_o R} dv'$$

$$\rho_{ps} = \mathbf{P} \cdot \mathbf{a}_n$$
$$\rho_{pv} = -\nabla \cdot \mathbf{P}$$

The total positive bound charge on surface S bounding the dielectric is

$$Q_b = \oint \mathbf{P} \cdot d\mathbf{S} = \int \rho_{ps}\, dS$$

while the charge that remains inside S is

$$-Q_b = \int_v \rho_{pv}\, dv = -\int_v \nabla \cdot \mathbf{P}\, dv$$

Thus the total charge of the dielectric material remains zero, that is,

$$\text{Total charge} = \oint_S \rho_{ps}\, dS + \int_v \rho_{pv}\, dv = Q_b - Q_b = 0$$

We now consider the case in which the dielectric region contains free charge. If p_v is the free charge volume density, the total volume charge density p_t is given by

$$\rho_t = \rho_v + \rho_{pv} = \nabla \cdot \varepsilon_o \mathbf{E}$$

$$\rho_v = \nabla \cdot \varepsilon_o \mathbf{E} - \rho_{pv}$$
$$= \nabla \cdot (\varepsilon_o \mathbf{E} + \mathbf{P})$$
$$= \nabla \cdot \mathbf{D}$$

$$\boxed{\mathbf{D} = \varepsilon_o \mathbf{E} + \mathbf{P}}$$

We would expect that the polarization P would vary directly as the applied electric field E. For some dielectrics, this is usually the case and we have

$$\boxed{\mathbf{P} = \chi_e \varepsilon_o \mathbf{E}}$$

DIELECTRIC CONSTANT AND STRENGTH

$$\mathbf{D} = \varepsilon_o(1 + \chi_e)\,\mathbf{E} = \varepsilon_o\varepsilon_r\mathbf{E}$$

$$\boxed{\mathbf{D} = \varepsilon\mathbf{E}}$$

$$\boxed{\varepsilon = \varepsilon_0\varepsilon_r}$$

$$\boxed{\varepsilon_r = 1 + \chi_e = \frac{\varepsilon}{\varepsilon_o}}$$

ε is called the *permittivity* of the dielectric, ε_o is the permittivity of free space, as approximately $10^{-9}/36\pi$ F/m, and ε_r is called the *dielectric constant* or *relative permittivity*.

The dielectric strength is the maximum electric field that a dielectric can tolerate or withstand without breakdown.

A dielectric material is linear if ε does not change with applied E field. homogeneous if ε does not change from point to point, and isotropic if ε does not change with direction.

$$\mathbf{F} = \frac{Q_1 Q_2}{4\pi\varepsilon_o\varepsilon_r R^2}\mathbf{a}_R$$

$$W = \frac{1}{2}\int \varepsilon_o\varepsilon_r E^2 \, dv$$

BOUNDARY CONDITIONS

Dielectric-Dielectric Boundary Conditions

Consider the E field existing in a region consisting of two different dielectrics characterized by $\varepsilon_1 = \varepsilon_o \varepsilon_{r1}$ and $\varepsilon_2 = \varepsilon_o \varepsilon_{r2}$ as shown in Figure. E_1 and E_2 in media 1 and 2, respectively, can be decomposed as

$$\mathbf{E}_1 = \mathbf{E}_{1t} + \mathbf{E}_{1n}$$

$$\mathbf{E}_2 = \mathbf{E}_{2t} + \mathbf{E}_{2n}$$

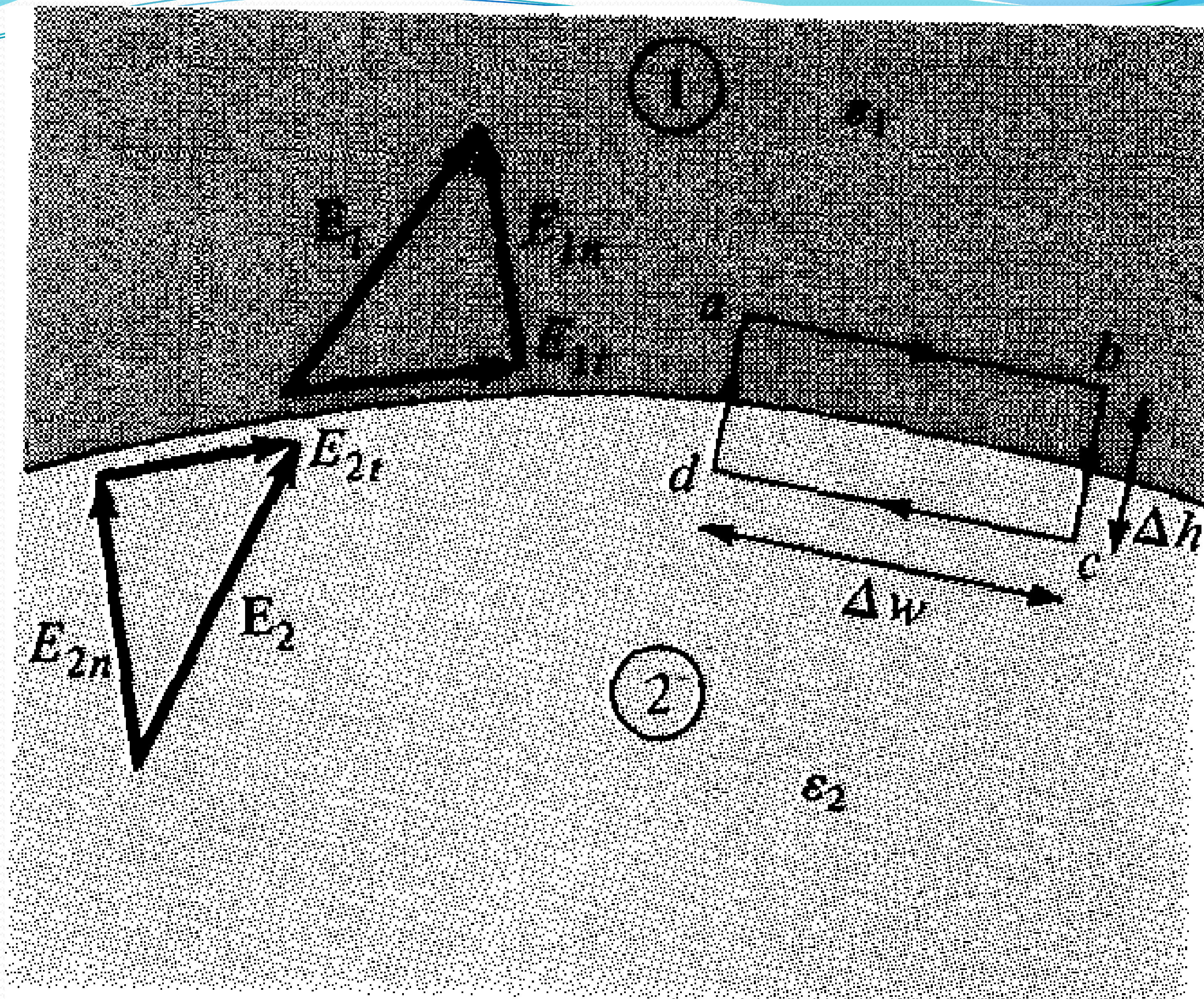
$$0 = E_{1t}\,\Delta w - E_{1n}\frac{\Delta h}{2} - E_{2n}\frac{\Delta h}{2} - E_{2t}\,\Delta w + E_{2n}\frac{\Delta h}{2} + E_{1n}\frac{\Delta h}{2}$$

$E_t = |\mathbf{E}_t|$ and $E_n = |\mathbf{E}_n|$. As $\Delta h \to 0$,

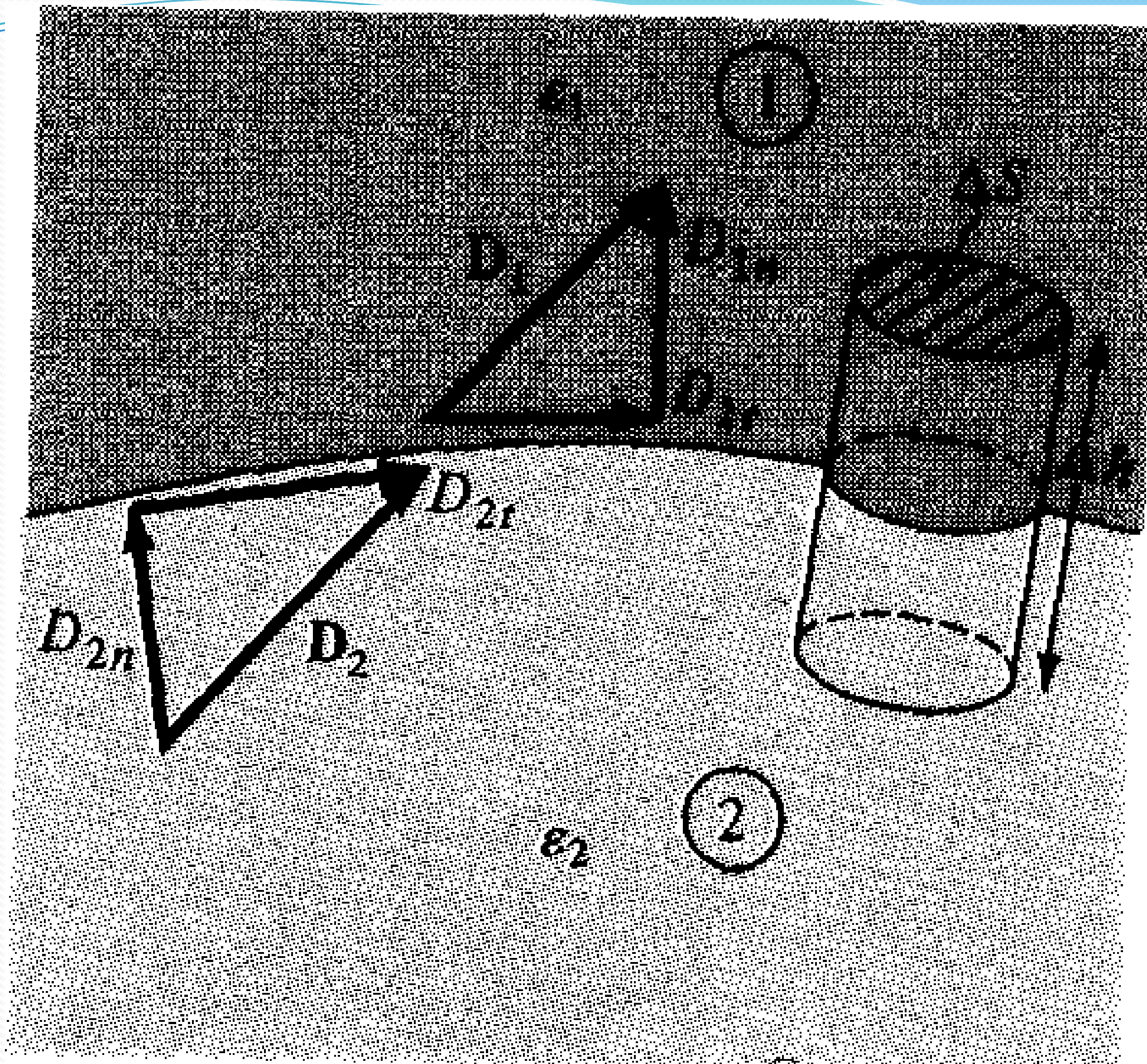
$$\boxed{E_{1t} = E_{2t}}$$

$$\frac{D_{1t}}{\varepsilon_1} = E_{1t} = E_{2t} = \frac{D_{2t}}{\varepsilon_2}$$

$$\frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}$$



1
ε_1
$\mathbf{E}_1$
E_{1n}
E_{1t}
a
b
E_{2t}
d
Δh
c
Δw
$\mathbf{E}_2$
E_{2n}
2
ε_2



1
D1
D1n
D1t
ΔS
Δh
D2t
D2n
D2
ε2
2

$$\Delta Q = \rho_S \Delta S = D_{1n} \Delta S - D_{2n} \Delta S$$

$$\boxed{D_{1n} - D_{2n} = \rho_S}$$

$$\boxed{D_{1n} = D_{2n}}$$

$$\varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}$$

$$E_1 \sin\theta_1 = E_{1t} = E_{2t} = E_2 \sin\theta_2$$

$$E_1 \sin\theta_1 = E_2 \sin\theta_2$$

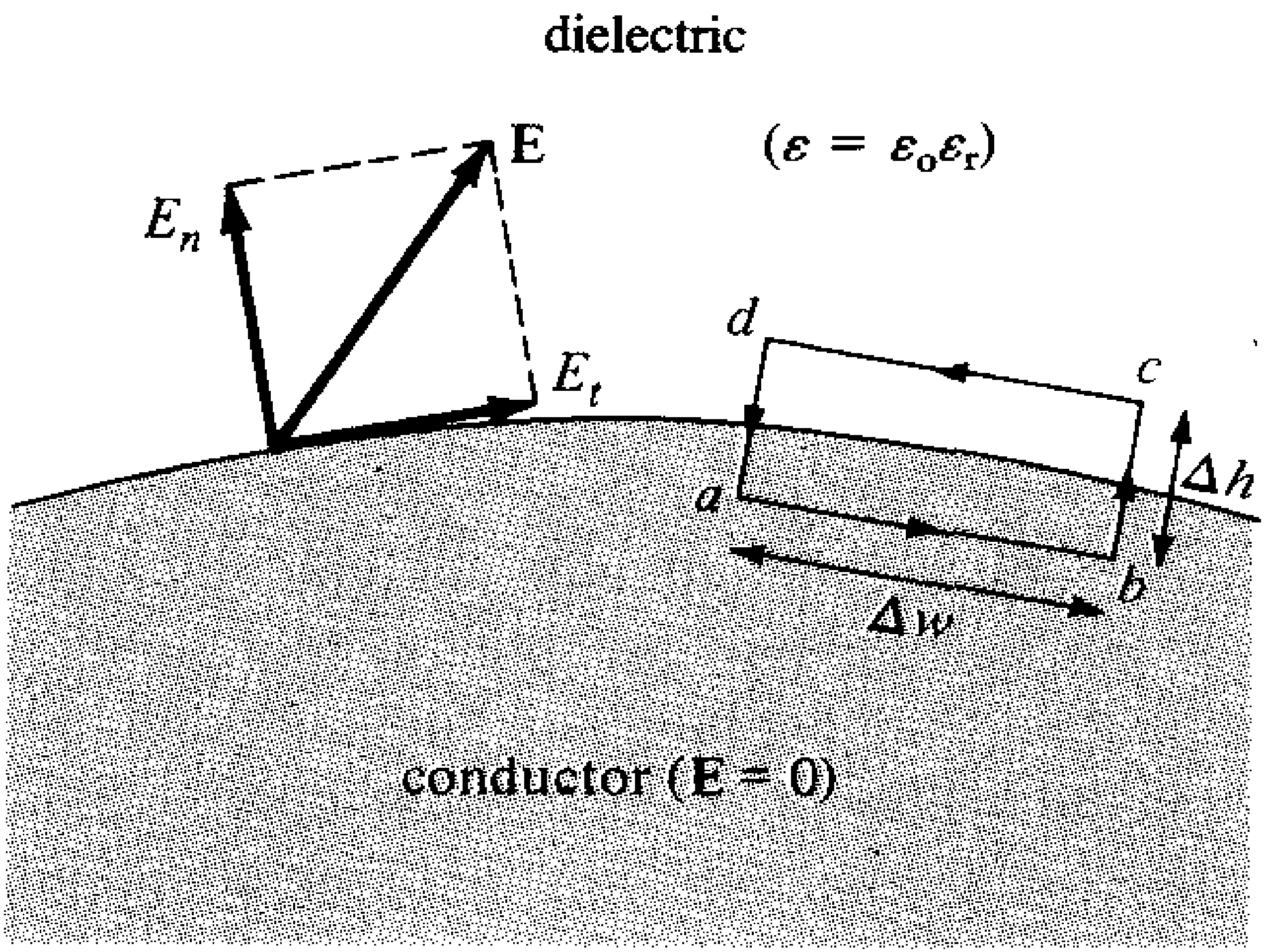
$$\varepsilon_1 E_1 \cos\theta_1 = D_{1n} = D_{2n} = \varepsilon_2 E_2 \cos\theta_2$$

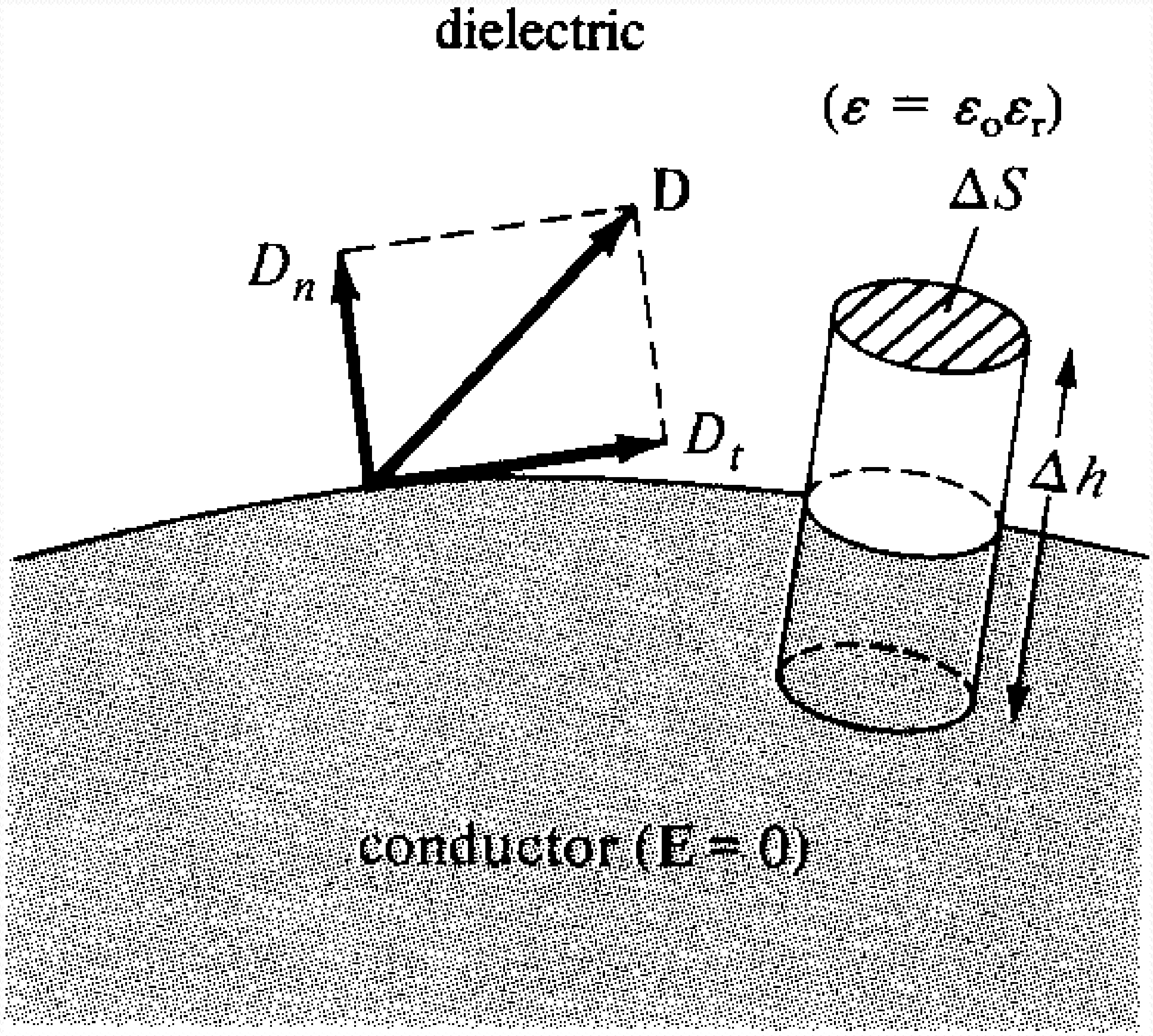
$$\varepsilon_1 E_1 \cos\theta_1 = \varepsilon_2 E_2 \cos\theta_2$$

$$\frac{\tan\theta_1}{\varepsilon_1} = \frac{\tan\theta_2}{\varepsilon_2}$$

$$\boxed{\frac{\tan\theta_1}{\tan\theta_2} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}}$$

Conductor-Dielectric Boundary





dielectric
$(\varepsilon = \varepsilon_o \varepsilon_r)$
$\mathbf{D}$
D_n
D_t
ΔS
Δh
conductor ($\mathbf{E} = 0$)

$$0 = 0 \cdot \Delta w + 0 \cdot \frac{\Delta h}{2} + E_n \cdot \frac{\Delta h}{2} - E_t \cdot \Delta w - E_n \cdot \frac{\Delta h}{2} - 0 \cdot \frac{\Delta h}{2}$$

As $\Delta h \to 0$,

$$E_t = 0$$

$$\Delta Q = D_n \cdot \Delta S - 0 \cdot \Delta S$$

$$D_n = \frac{\Delta Q}{\Delta S} = \rho_S$$

$$D_n = \rho_S$$

1. No electric field may exist *within* a conductor; that is,

$$\boxed{\rho_v = 0, \qquad \mathbf{E} = 0} \tag{5.70}$$

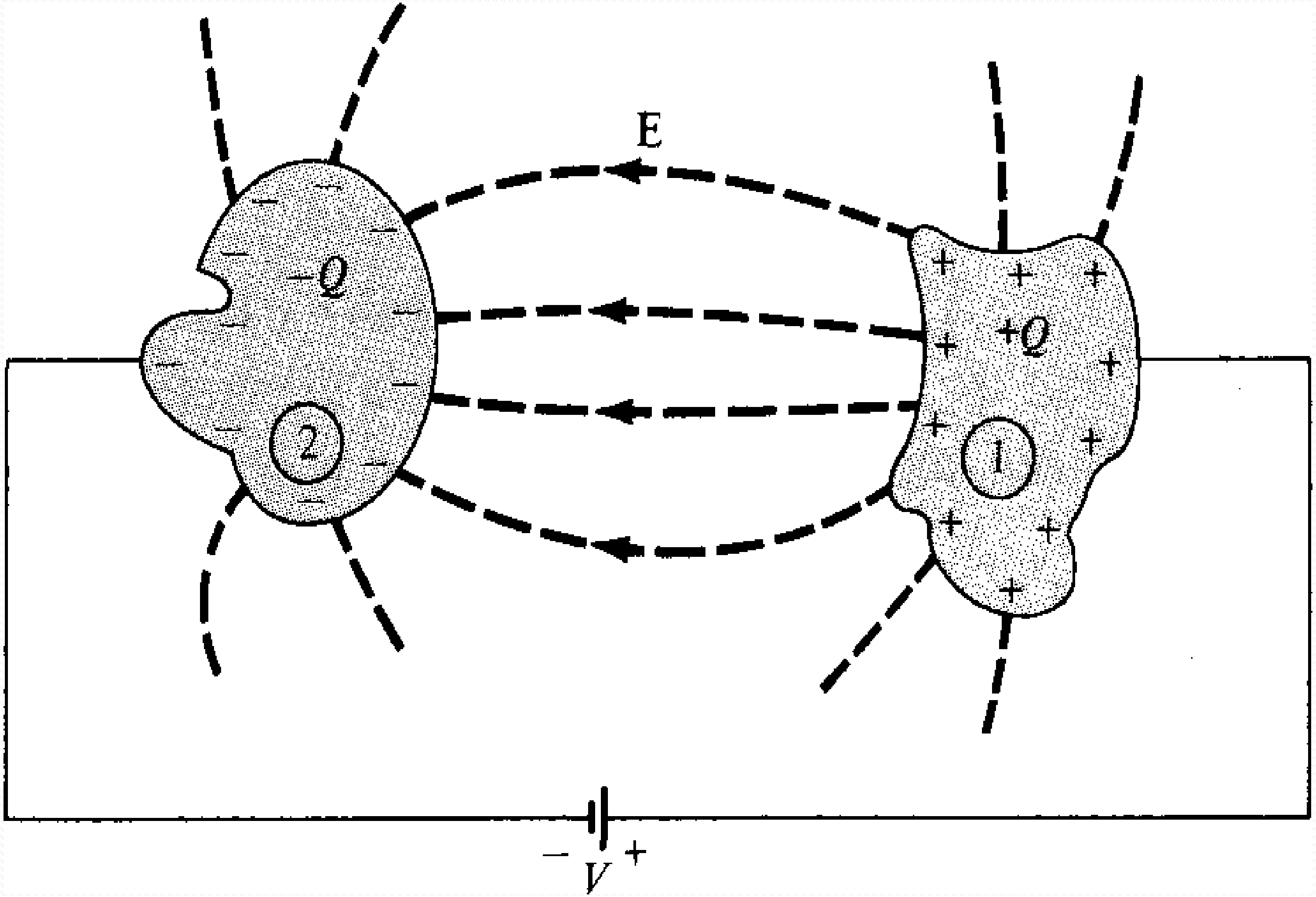
2. Since $\mathbf{E} = -\nabla V = 0$, there can be no potential difference between any two points in the conductor; that is, a conductor is an equipotential body.
3. The electric field **E** can be external to the conductor and *normal* to its surface; that is

$$\boxed{D_t = \varepsilon_o \varepsilon_r E_t = 0, \qquad D_n = \varepsilon_o \varepsilon_r E_n = \rho_S} \tag{5.71}$$

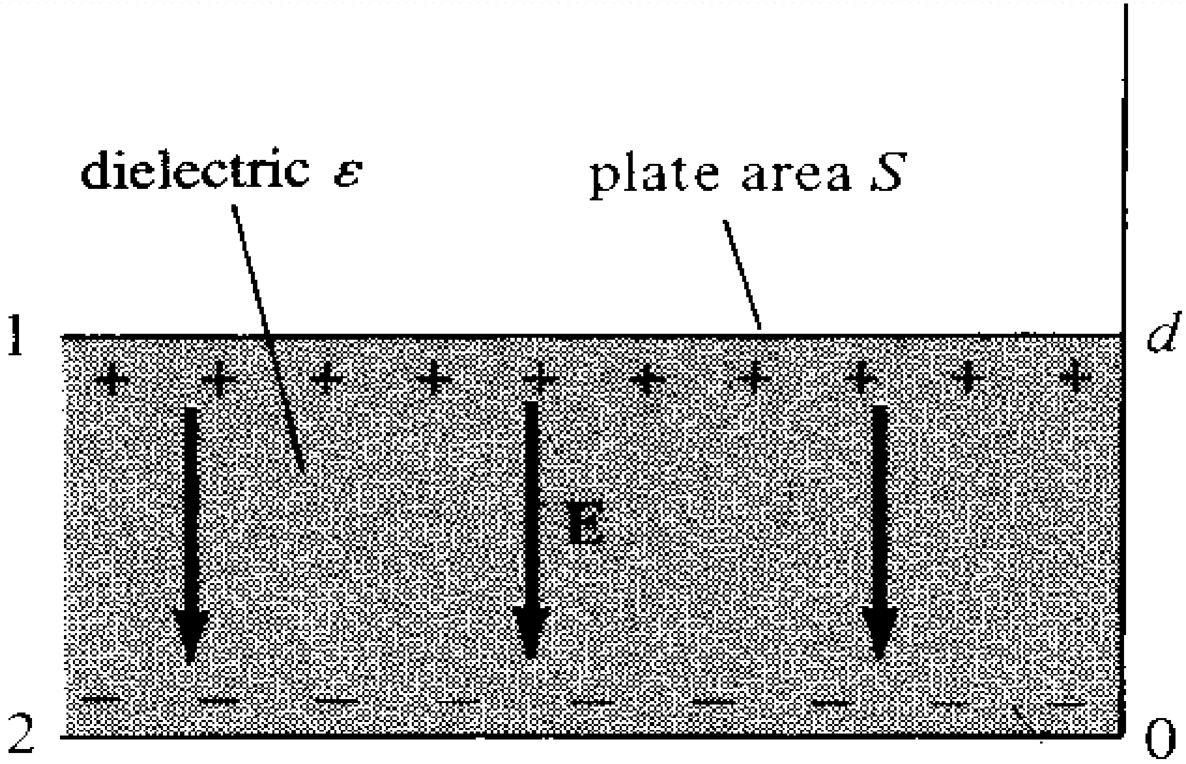
Capacitance

$$C = \frac{Q}{V} = \frac{\varepsilon \oint \mathbf{E} \cdot d\mathbf{S}}{\int \mathbf{E} \cdot d\mathbf{l}}$$

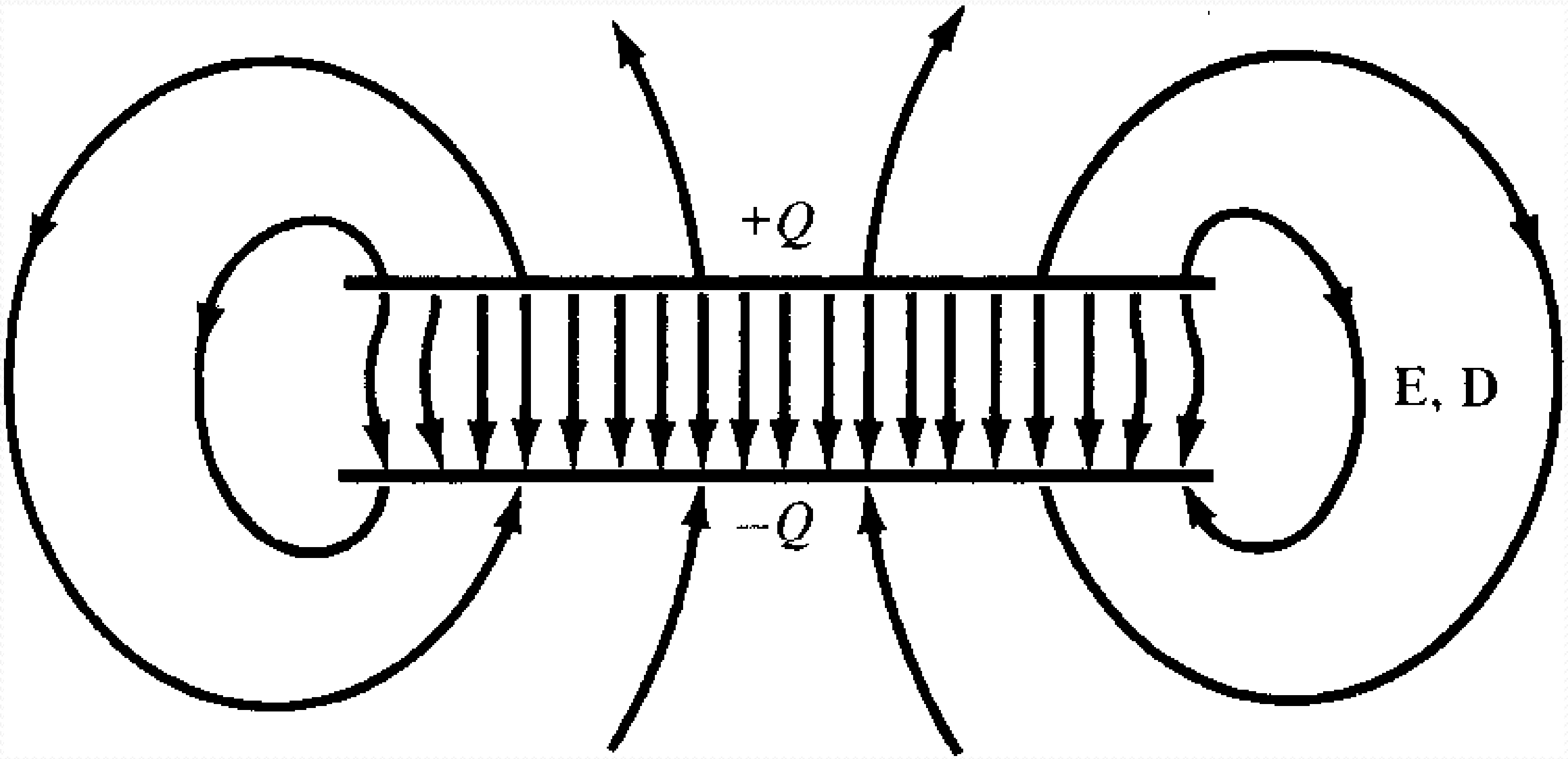
$$V = V_1 - V_2 = -\int_2^1 \mathbf{E} \cdot d\mathbf{l}$$


E
−Q
+Q
2
1
−
V
+

Parallel-Plate Capacitor



$$\rho_S = \frac{Q}{S}$$


+Q
E, D
−Q

$$\mathbf{E} = \frac{\rho_S}{\varepsilon}(-\mathbf{a}_x)$$

$$= -\frac{Q}{\varepsilon S}\mathbf{a}_x$$

$$V = -\int_2^1 \mathbf{E}\cdot d\mathbf{l} = -\int_0^d \left[-\frac{Q}{\varepsilon S}\mathbf{a}_x\right]\cdot dx\,\mathbf{a}_x = \frac{Qd}{\varepsilon S}$$

$$\boxed{C = \frac{Q}{V} = \frac{\varepsilon S}{d}}$$

$$\varepsilon_r = \frac{C}{C_o}$$

$$\boxed{W_E = \frac{1}{2}CV^2 = \frac{1}{2}QV = \frac{Q^2}{2C}}$$

$$W_E = \frac{1}{2}\int_v \varepsilon \frac{Q^2}{\varepsilon^2 S^2}\, dv = \frac{\varepsilon Q^2 S d}{2\varepsilon^2 S^2}$$

$$= \frac{Q^2}{2}\left(\frac{d}{\varepsilon S}\right) = \frac{Q^2}{2C} = \frac{1}{2}QV$$

Coaxial Capacitor

$$Q = \varepsilon \oint \mathbf{E} \cdot d\mathbf{S} = \varepsilon E_\rho 2\pi\rho L$$

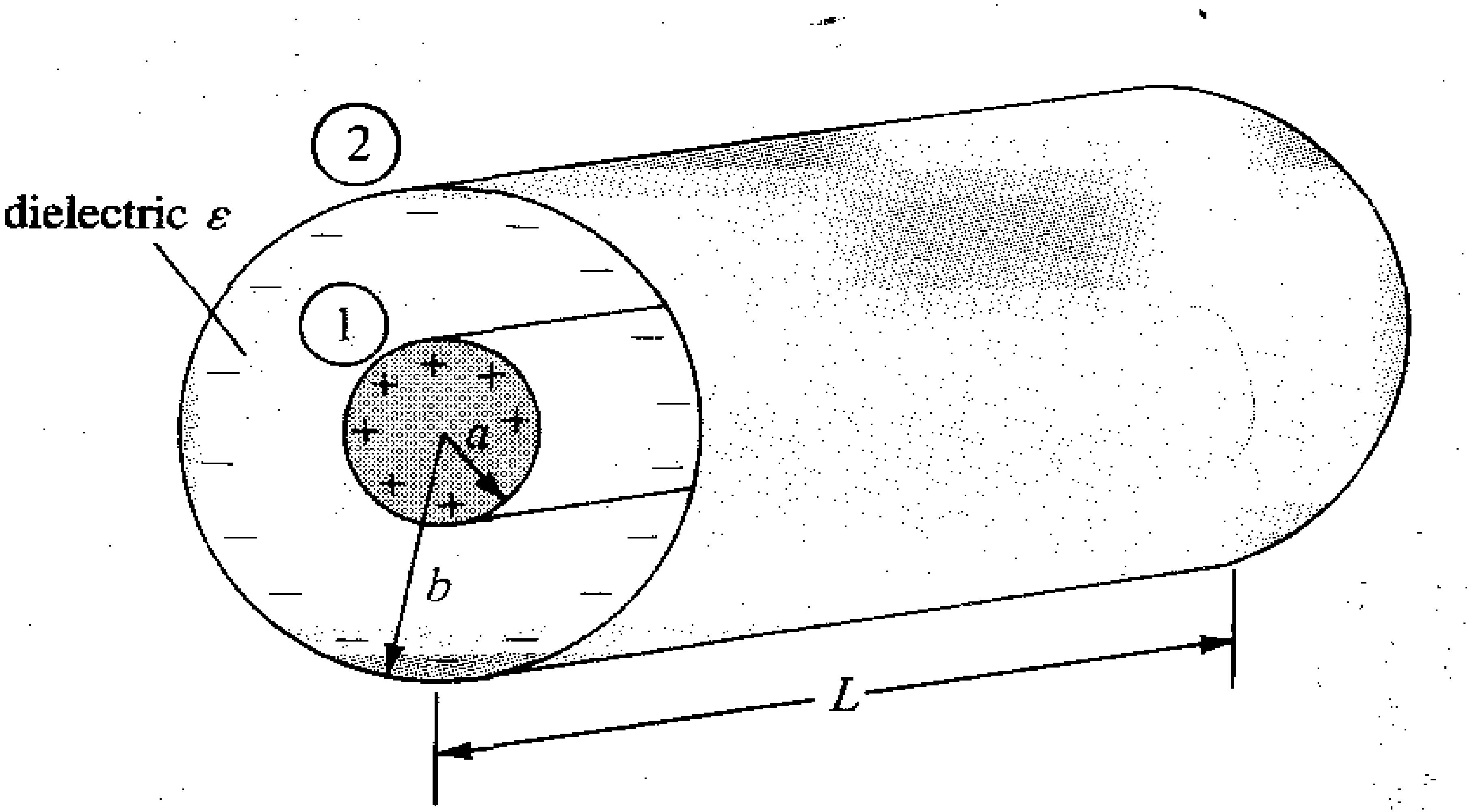
$$\mathbf{E} = \frac{Q}{2\pi\varepsilon\rho L}\mathbf{a}_\rho$$

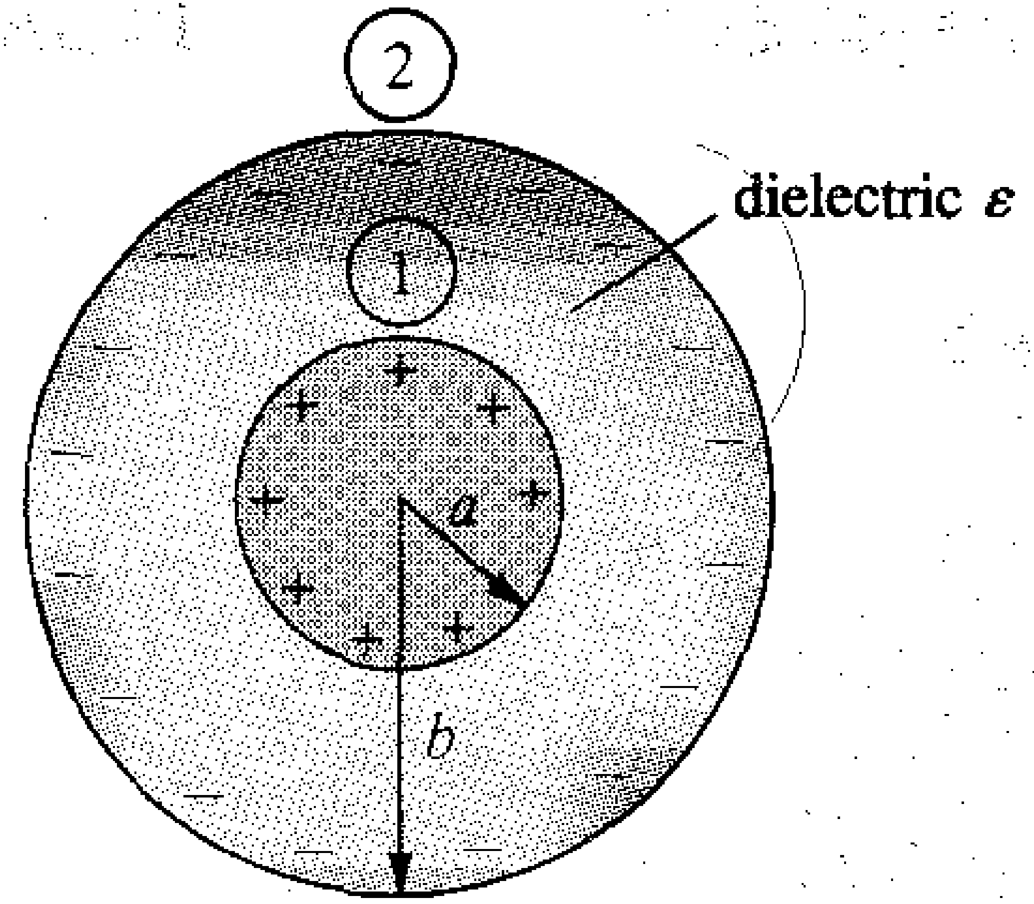
$$V = -\int_2^1 \mathbf{E} \cdot d\mathbf{l} = -\int_b^a \left[\frac{Q}{2\pi\varepsilon\rho L}\mathbf{a}_\rho\right] \cdot d\rho\, \mathbf{a}_\rho$$

$$= \frac{Q}{2\pi\varepsilon L}\ln\frac{b}{a}$$

$$C = \frac{Q}{V} = \frac{2\pi\varepsilon L}{\ln\frac{b}{a}}$$

Spherical Capacitor




2
1
dielectric ε
a
b

$$Q = \varepsilon \oint \mathbf{E} \cdot d\mathbf{S} = \varepsilon E_r 4\pi r^2$$

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon r^2}\mathbf{a}_r$$

$$V = -\int_2^1 \mathbf{E} \cdot d\mathbf{l} = -\int_b^a \left[\frac{Q}{4\pi\varepsilon r^2}\mathbf{a}_r\right] \cdot dr\,\mathbf{a}_r$$

$$= \frac{Q}{4\pi\varepsilon}\left[\frac{1}{a} - \frac{1}{b}\right]$$

$$C = \frac{Q}{V} = \frac{4\pi\varepsilon}{\frac{1}{a} - \frac{1}{b}}$$

Fundamental Principles and Relationships

$$\text{Poisson}: \nabla^2 \mathrm{V} = -\frac{\rho}{\epsilon_0} \quad \text{Laplace}: \nabla^2 \mathrm{V} = 0 \tag{1}$$

$$\text{Cylindrical}: \nabla^2 \mathrm{V} = \frac{1}{\mathrm{s}}\frac{\partial}{\partial \mathrm{s}}\left(\mathrm{s}\frac{\partial \mathrm{V}}{\partial \mathrm{s}}\right) \tag{2}$$

$$+\frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} \tag{3}$$

$$+\frac{\partial^2 V}{\partial z^2} \tag{4}$$

$$a) \quad \frac{d^2V}{ds^2} = 0 \qquad b) \quad \frac{d}{ds}(s\frac{dV}{ds}) = 0 \tag{5}$$

$$c) \quad \frac{d^2V}{ds^2} = -\frac{\rho_0}{\epsilon_0} \qquad d) \quad \frac{d}{ds}(s\frac{dV}{ds}) = -\frac{\rho_0}{\epsilon_0} \tag{6}$$

Current And Current Density

$$I = \frac{dQ}{dt}$$

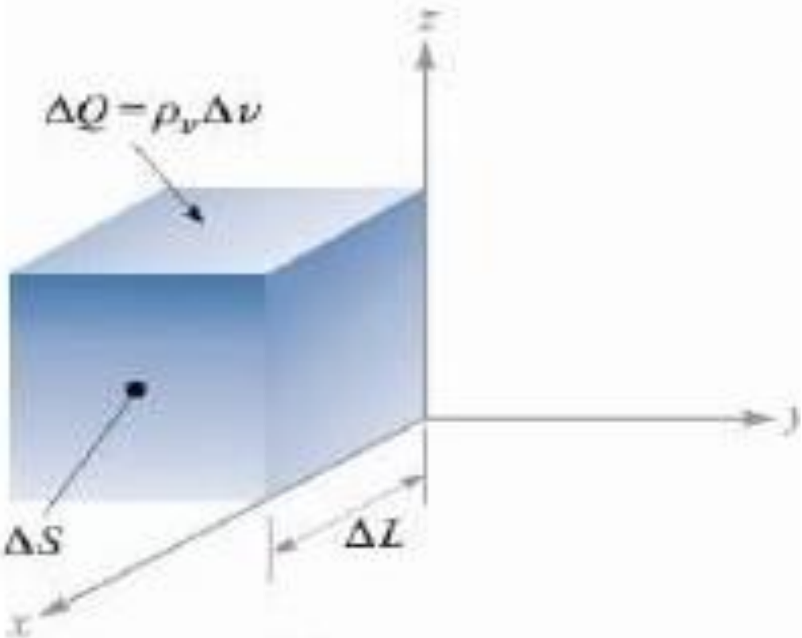
$$\Delta I = J_N \Delta S$$

$$\Delta I = \mathbf{J} \cdot \Delta \mathbf{S}$$

$$I = \int_S \mathbf{J} \cdot d\mathbf{S}$$

$$\Delta I = \frac{\Delta Q}{\Delta t} = \rho_v \, \Delta S \, \frac{\Delta x}{\Delta t}$$

$$\Delta I = \rho_v \, \Delta S \, v_x$$



(a)

$$J_x = \rho_v \, v_x$$

$$\boxed{\mathbf{J} = \rho_v \mathbf{v}}$$

Continuity of Current

$$I = \oint_S \mathbf{J} \cdot d\mathbf{S} = -\frac{dQ_i}{dt}$$

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = \int_{\text{vol}} (\nabla \cdot \mathbf{J})\, dv$$

$$\int_{\text{vol}} (\nabla \cdot \mathbf{J})\, dv = -\frac{d}{dt}\int_{\text{vol}} \rho_v\, dv$$

$$\int_{\text{vol}} (\nabla \cdot \mathbf{J})\, dv = \int_{\text{vol}} -\frac{\partial \rho_v}{\partial t}\, dv$$

$$(\nabla \cdot \mathbf{J})\, \Delta v = -\frac{\partial \rho_v}{\partial t}\, \Delta v$$

$$\boxed{(\nabla \cdot \mathbf{J}) = -\frac{\partial \rho_v}{\partial t}}$$

Resistance & Ohm's Law

$$\mathbf{F} = -e\mathbf{E}$$

$$\mathbf{v}_d = -\mu_e \mathbf{E}$$

$$\mathbf{J} = -\rho_e \mu_e \mathbf{E}$$

$$\mathbf{J} = \sigma \mathbf{E}$$

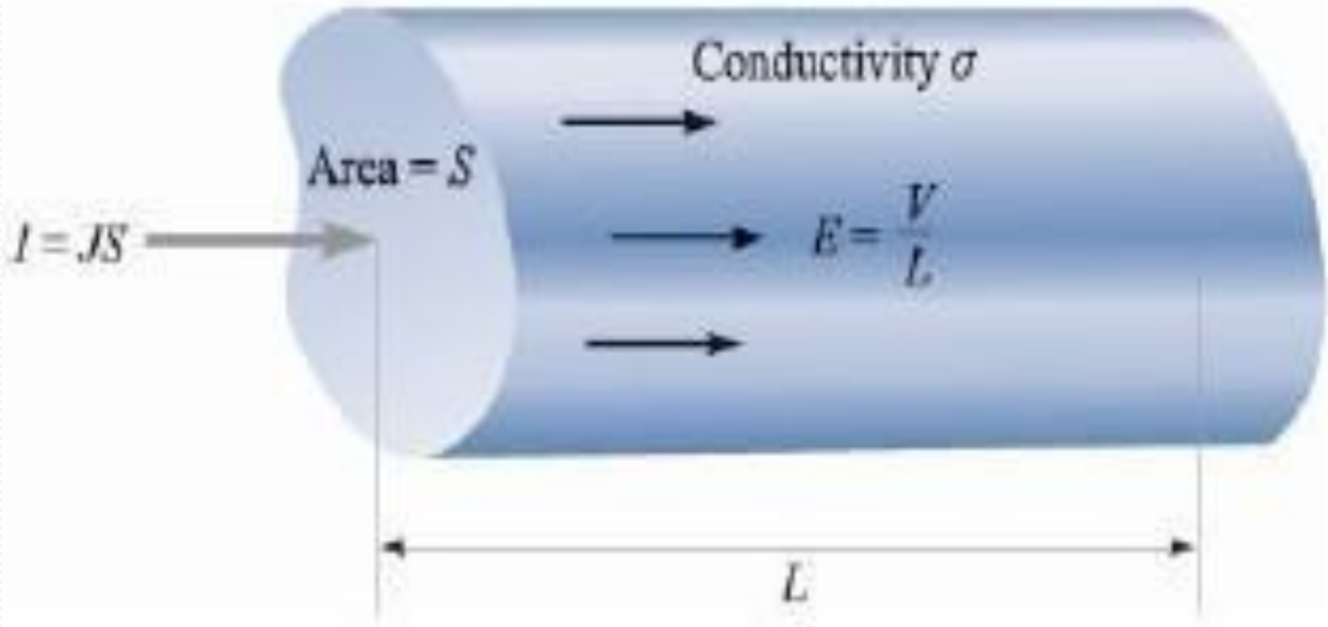
$$\sigma = -\rho_e \mu_e$$

$$I = \int_S \mathbf{J} \cdot d\mathbf{S} = J\,S$$

$$V_{ab} = -\int_b^a \mathbf{E} \cdot d\mathbf{L} = -\mathbf{E} \cdot \int_b^a d\mathbf{L} = -\mathbf{E} \cdot \mathbf{L}_{ba}$$

$$= \mathbf{E} \cdot \mathbf{L}_{ab}$$

$$V = EL$$



$$J = \frac{I}{S} = \sigma E = \sigma \frac{V}{L}$$

$$V = \frac{L}{\sigma S} I$$

$$V = IR$$

$$R = \frac{L}{\sigma S}$$

$$R = \frac{V_{ab}}{I} = \frac{-\int_b^a \mathbf{E} \cdot d\mathbf{L}}{\int_S \sigma \mathbf{E} \cdot d\mathbf{S}}$$

Unit III
MAGNETOSTATICS

Force on a Charged Particle

The electric force *Fe* on a stationary or moving electric charge *Q* in an electric field is given by Coulomb's experimental law and is related to the electric field intensity E as

$$F_e = QE$$

This shows that if F_Q is positive, F and E have the same direction.

A magnetic field can exert force only on a moving charge.

From experiments, it is found that the magnetic force F_m experienced by a charge Q moving with a velocity u in a magnetic field B is

$$F_m = Qu \: X \: B$$

F_m is perpendicular to both u and B.

Lorentz Force equation

For a moving charge Q in the presence of both electric and magnetic fields, the total force on the charge is given by

$$F = Fe + Fm$$

Or

$$F = Q(E + u \times B)$$

This equation is known as Lorentz Force equation.

It relates mechanical force to electrical force.

If the mass of the charged particle moving in E and B fields is *m*, by Newton's second law of motion.

$$F = m\frac{du}{dt} = Q\left(E + u X B\right)$$

The solution to this equation is important in determining the motion of charged particles in E and B fields.

Force on a Current Element

To determine the force on a current element Idl of a current-carrying conductor due to the magnetic field B

$$J = \rho_v u$$

We know that

$$I\, dl = K\, dS = J\, dv$$

Then

$$I\, dl = \rho_u\, dv = dQ\, u$$

Hence

$$I\,dl = dQ\,u$$

The force acting on an elemental charge dQ moving with velocity u is equivalent to a conduction current element I dl in a magnetic field B.

$$dF = I\,dl\,X\,B$$

If the current is through a closed path L or circuit, the force on circuit is given by

$$F = \oint I\,dl\,X B$$

The magnetic field produced by the current element I*dl* does not exert force on the element itself just as a point charge does not exert force on itself.

The B field that exerts force on I dl $_l$ must be due to another element.

If instead of the line current element I dl$_{l,}$ we have surface current elements KdS or a volume current element J dv, Then

$$dF = K\,dS\ X\ B \quad \text{Or} \quad dF = J\ dv\ X B$$

Then

$$F = \oint K \, dS X B$$

and

$$F = \oint J \, dv X B$$

Ampere' Circuital Law

- Ampere'scircuit law states that the line integral of the tangential component of H around a dosed path is the same as the net current I_{enc}.IK. enclosed by the path.

$$dH_z = -\frac{nI}{2} \sin\theta \, d\theta$$

$$H_z = -\frac{nI}{2} \int^{\theta_2} \sin\theta \, d\theta$$

$$\mathbf{H} = \frac{nI}{2} (\cos\theta_2 - \cos\theta_1) \, \mathbf{a}_z$$

Substituting $n = N/\ell$ gives

$$\mathbf{H} = \frac{NI}{2\ell} (\cos\theta_2 - \cos\theta_1) \, \mathbf{a}_z$$

At the center of the solenoid,

$$\cos\theta_2 = \frac{\ell/2}{[a^2 + \ell^2/4]^{1/2}} = -\cos\theta_1$$

$$\mathbf{H} = \frac{In\ell}{2[a^2 + \ell^2/4]^{1/2}}\,\mathbf{a}_z$$

If $\ell \gg a$ or $\theta_2 \simeq 0°$, $\theta_1 \simeq 180°$,

$$\mathbf{H} = nI\mathbf{a}_z = \frac{NI}{\ell}\,\mathbf{a}_z$$

Ampere's Circuital Law

Ampere's circuit law states that the line

- integral of the tangential component of H
- around a closed path is the same as the net

current I enclosed by the path.

- enc

In other words, the circulation of **H** equals I_{enc}; that is,

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{enc}$$

$$I_{enc} = \oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S}$$

$$I_{enc} = \int_S \mathbf{J} \cdot d\mathbf{S}$$

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J}}$$

FORCES DUE TOMAGNETIC FIELDS

- There are at least three ways in which force due to magnetic fields can be experienced.

1. Due to a moving charged particle in a B field.
2. On a current element in an external B field.
3. Between two current elements.

Force between Two Current Elements

Let us now consider the force between two elements $I_1\ dl_1$ and $I_2\ dl_2$.

According to Biot-Savart's law, both current elements produce magnetic fields.

So we may find the force $d(dF_1)$ on element $I_1\ dl_1$ due to the field $dB2$ produced by element $I_2\ dl_2$

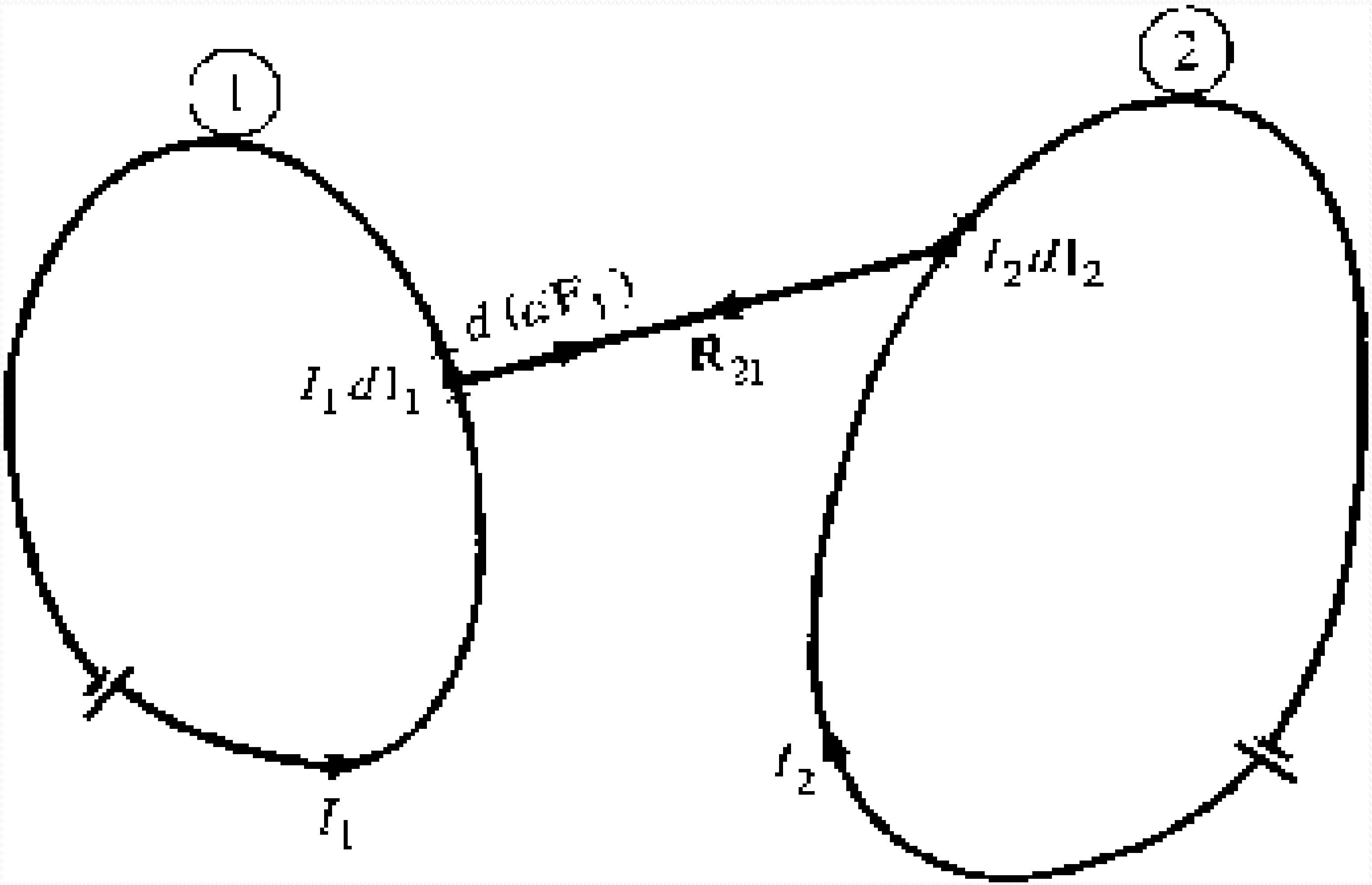
$$d(dF_1) = I_1 dl_1 X\ dB_2$$

From Biot-Savart's law

$$dB_2 = \frac{\mu_0 I_2 \, dI_2 \, X \, a_{R_{21}}}{4\pi R_{21}^2}$$

$$d\left(dF_1\right) = \frac{\mu_0 I_1 \, dI_1 \, X \left(\mu_0 I_2 \, dI_2 \, X \, a_{R_{21}}\right)}{4\pi R_{21}^2}$$

$$F_1 = \frac{\mu_0 I_1 I_2}{4\pi} \oint\oint \frac{dI_1 \, X \left(dI_2 \, X a_{R_{21}}\right)}{R_{21}^2}$$

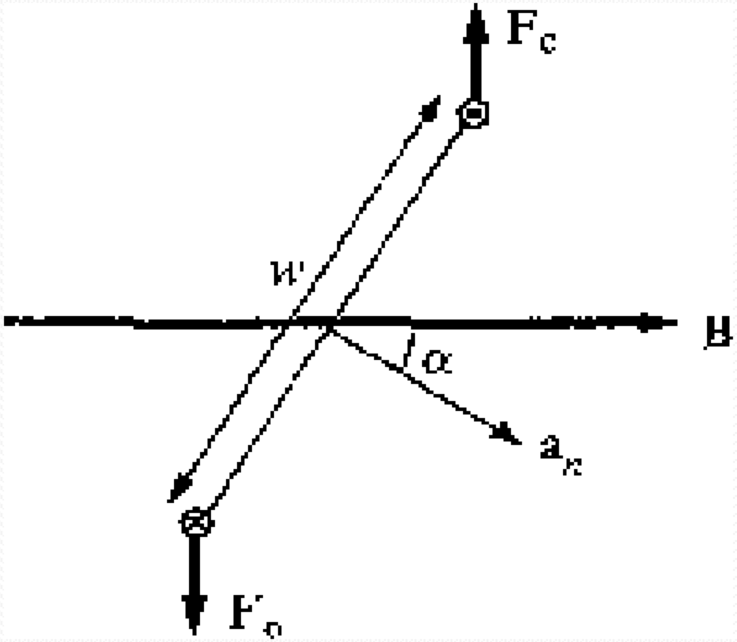


1
2
$d(dF_1)$
$I_1 d\mathbf{l}_1$
$\mathbf{R}_{21}$
$I_2 d\mathbf{l}_2$
I_1
I_2

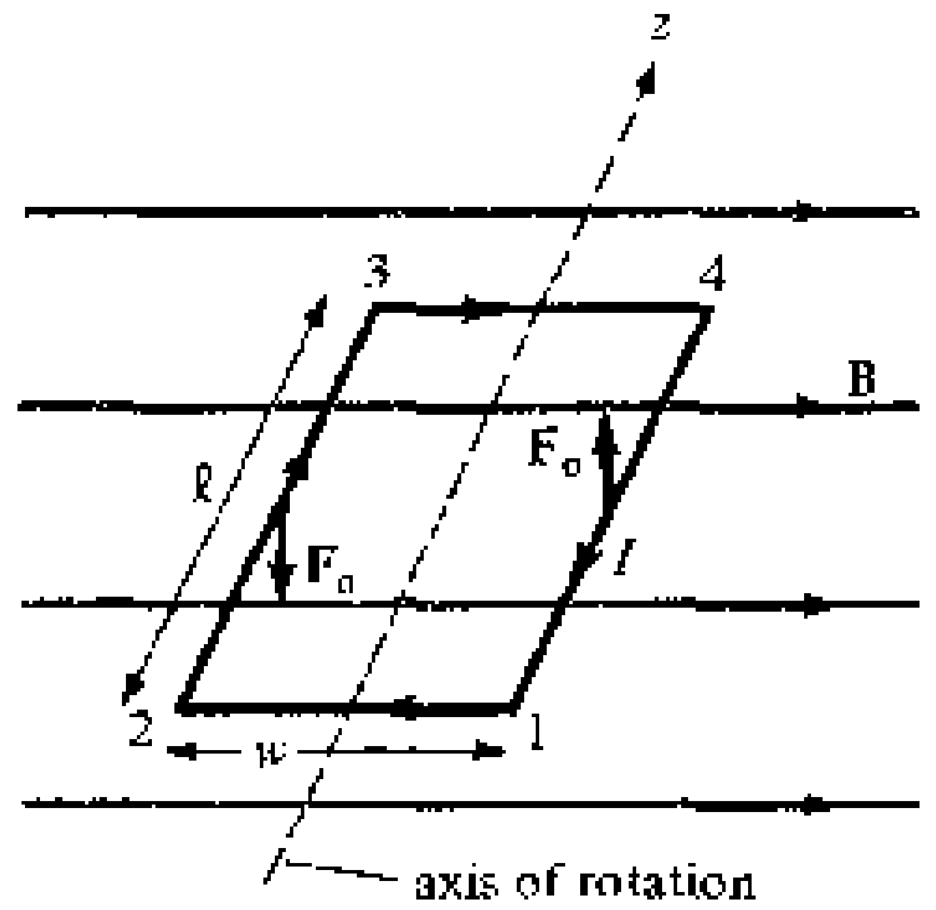
MAGNETIC TORQUE AND MOMENT

The **torque T** (or mechanical moincnl of force) on ihe loop is the vector product of the force F and the moment arm r.

That is T =r XF



Consider a rectangular loop of length l and width *w* placed in a uniform magnetic field B as shown in Figure



$$F = I \int_{2}^{3} dl \, X \, B \, \square \, I \int_{4}^{1} dl \, X \, B$$

$$F = I \int_{0}^{l} dz \, a_z \, X \, B \, \square \, I \int_{l}^{0} dz \, a_z \, X \, B$$

Or

$$F = F_0 - F_0 = 0$$

Thus F = Bil. Thus no force is exerted on the loop as a whole. However Fo and -Fo act at different points on the loop, there by creating a couple.

The torque on the loop is

$$|T| = |F_0| w \sin \theta$$

Or

$$T = BIlw \sin \theta$$

But lw = S, the area of the loop

$$T = BIS \sin \theta$$

We define the quantity

$$m = IS\, a_n$$

m is defined as the magnetic dipole moment.

Units are A/m².

Magnetic Dipole Moment

The **magnetic dipole moment** is the product of current and area of the loop.

Its direction is normal to the loop.

Torque on a magnetic loop placed in Magnetic field is

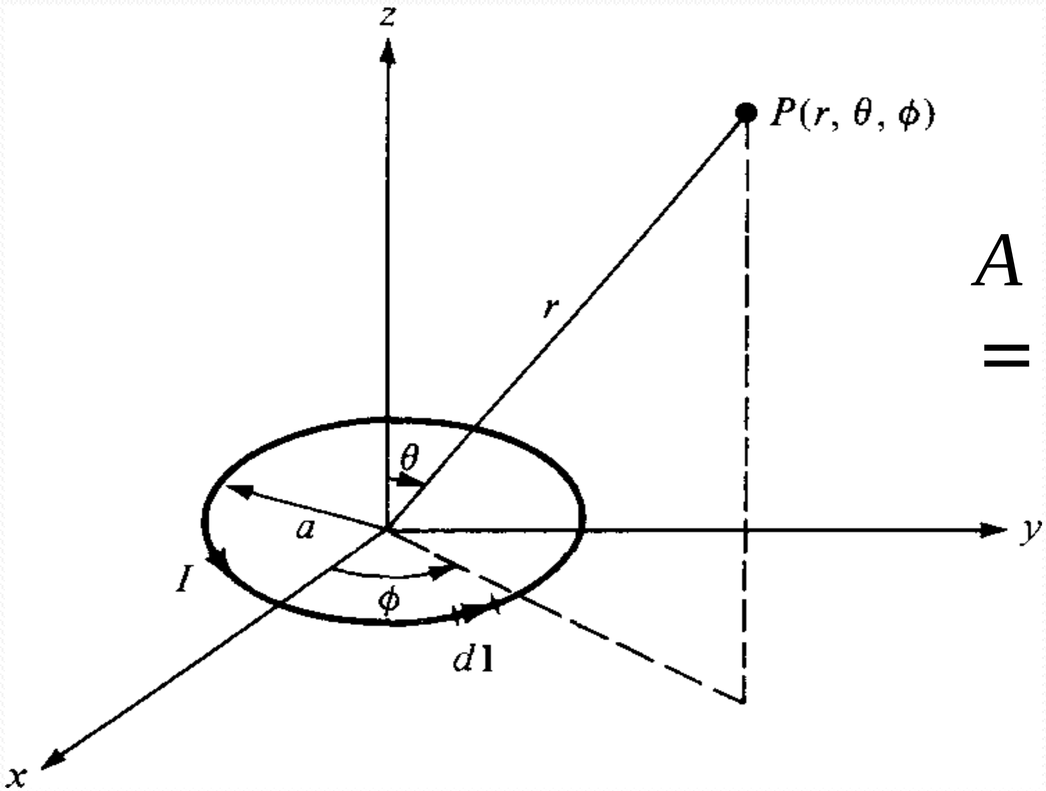
$$T = m \times B$$

This is applicable only when the magnetic field is uniform in nature.

Field due to a Magnetic Dipole

- A Bar magnet or a small filamentary current loop is usually referred to as a magnetic dipole.
- Consider a current carrying loop carrying a current of I amps, the the magnetic field due this at any arbitrary point $P(r,\theta,\varphi)$ due the loop is calculated as follows.

The magnetic Vector Potential at P is



$$A = \frac{\mu_0 I}{4} \oint \frac{dl}{r}$$

$$A = \frac{\mu_0 I \, a^2 \sin\theta \, a_\phi}{4 r^2}$$

or

$$A = \frac{\mu_0 m X a_r}{4 r^2}$$

The magnetic Flux density B is determined as

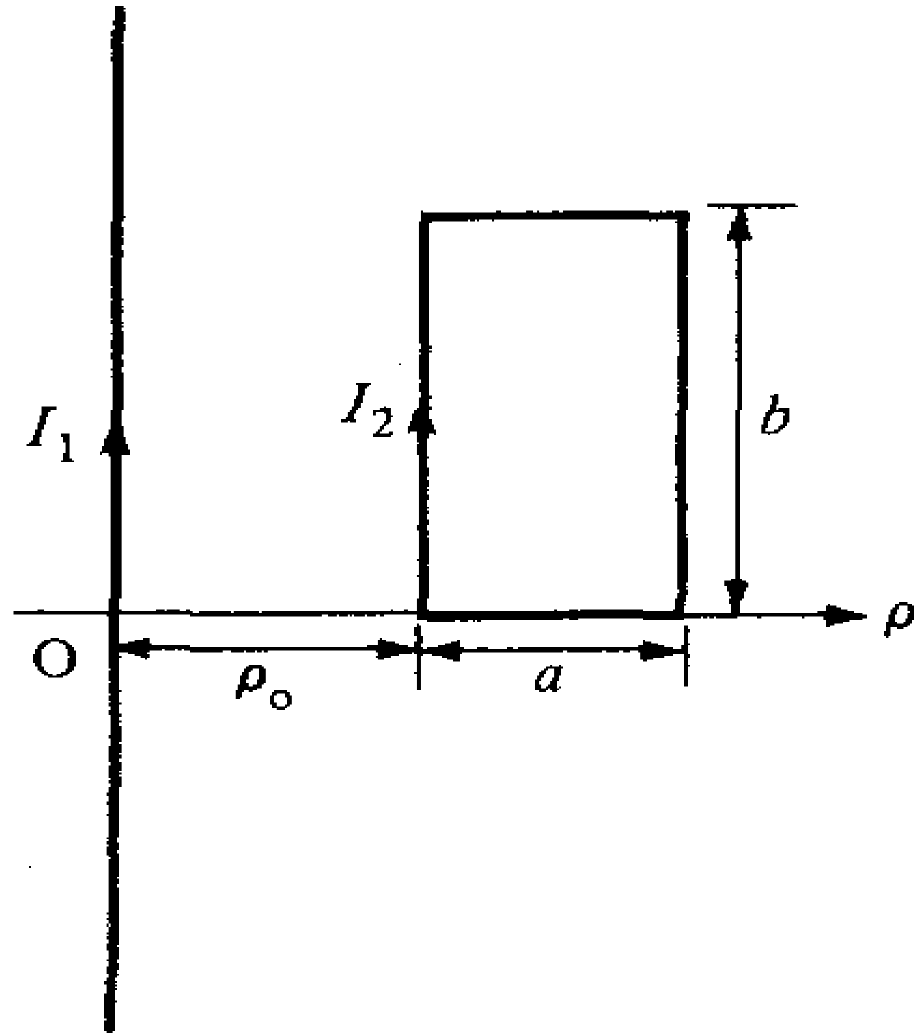
$$B = \nabla \times A$$

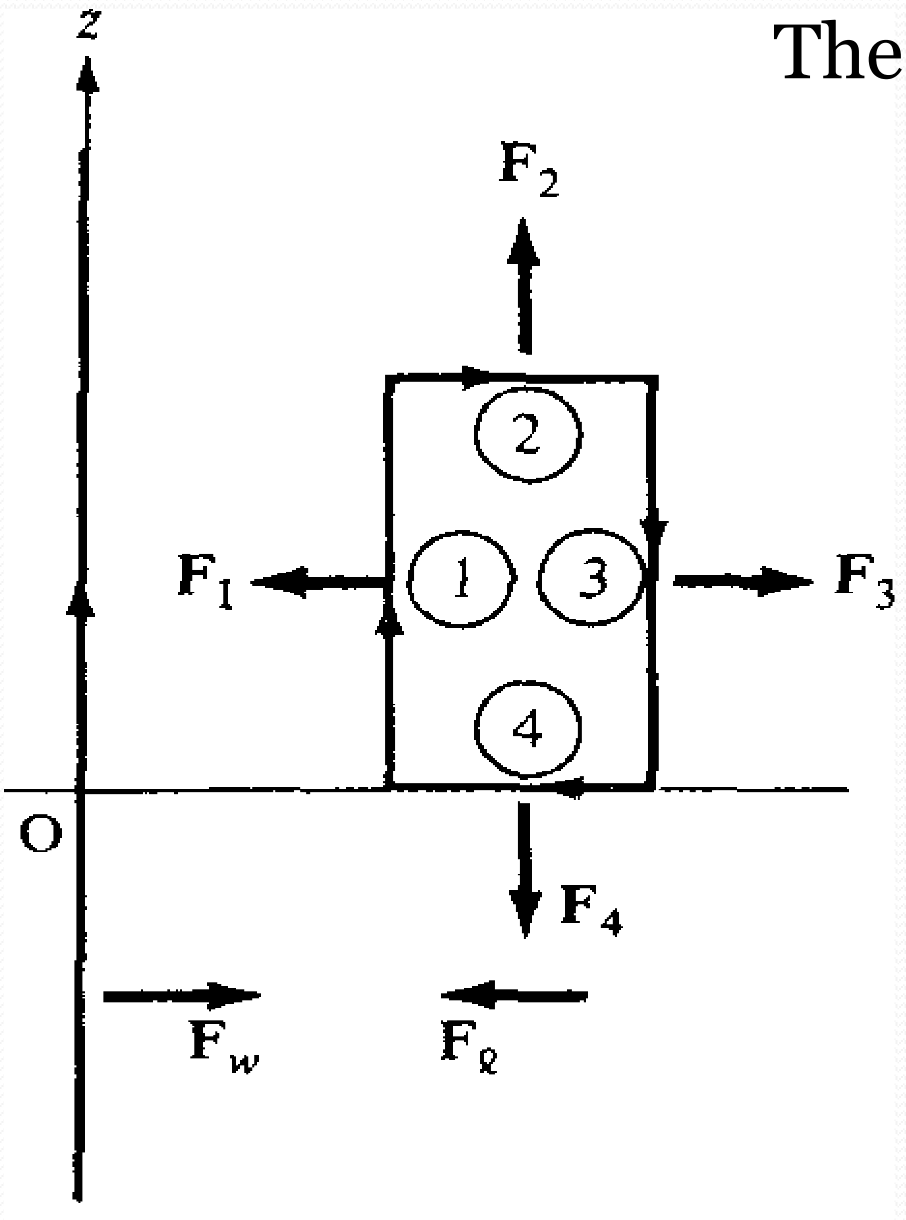
Therefore

$$B = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\, a_r + \sin\theta\, a_\theta\right)$$

Force Experienced

A rectangular loop carrying current I_2 is placed parallel to an infinitely long filamentary wire carrying current I_1 as shown in Figure





The force acting on loop is

$$F_l = F_1 + F_2 + F_3 + F_4$$

$$F_l = I_2 \oint dI_2\, X\, B$$

where F_1, F_2, F_3, and F_4 are respectively, the forces exerted on sides of the loop labeled 1, 2, 3, and 4 in Figure

Due to the infinitely long wire

$$B_1 = \frac{\mu_0 I_1}{2\pi\rho_0} a_\phi$$

Then

$$F_1 = I_2 \oint dI_2 \; X \; B_1$$

$$F_1 = I_2 \oint_{z=0}^{b} dz \, a_z \; X \; \frac{\mu_0 I_1}{2\pi\rho_0} a_\phi$$

$$F_1 = - \frac{\mu_0 I_1 I_2 b}{2\pi\rho_0} a_\rho$$ (Attractive)

$$F_3 = I_2 \oint dI_2 \times B_1$$

$$F_3 = I_2 \oint_{z=0} dz\, a_z \times \frac{\mu_0 I_1}{2\pi\rho_0} a_\phi$$

$$F_3 = \frac{\mu_0 I_1 I_2 b}{2\pi\rho_0} a_\rho \qquad \text{(Repulsive)}$$

$$F_2 = I_2 \oint_{\rho=\rho_0}^{\rho_0 + a} d\rho\, a_\rho \times \frac{\mu_0 I_1}{2\pi\rho_0} a_\phi$$

$$F_2 = \frac{\mu_0 I_1 I_2}{2\pi} \ln\left(\frac{\rho_0 + a}{\rho_0}\right) a_z \qquad \text{(Parallel)}$$

$=$

$$F_4 = I_2 \oint_{=\,0}^{0+a} d\,a \; X \frac{{}_0 I_1}{2\,\square a \square_0}\, a_{\square}$$

$$F_4 \; \frac{-\square I_1 I_2}{2} \ln \square \frac{{}_0 + a}{{}_0} \square a_z \quad \text{(Parallel)}$$

The Total Force

$$F_l = \frac{I_1 I_2 b}{2} \left[\frac{1}{{}_0} - \frac{1}{\square_0 + a} \right] \square a \; \square$$

Magnetization in Materials

- The magnetization M (in amperes/meter) is the magnetic dipole moment per unit volume.
- If there are N atoms in a given volume Δv and the k^{th} atom has a magnetic moment m_K.

$$M = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{N} m_k}{\Delta v}$$

- A medium for which M is not zero everywhere is said to be magnetized.

$$dA = \frac{□_0 M \quad X \, a_R}{4 \, R^2} \, dv^1$$

$$dA = \frac{□_0 M \quad X \, R}{4 \, R^3} \, dv^1$$

$$\frac{R}{R^3} = \nabla^1 \quad \frac{1}{R}$$

Hence

$$A = \frac{0}{4} \int M \;\; X \, \nabla^1 \quad \frac{1}{R} dv^1$$

But

$$M \, X \, \nabla^1 \frac{1}{R} = \frac{1}{R} \nabla^1 X \, M - \nabla^1 X \frac{M}{R}$$

Substituting the above equation in A

$$A = \frac{\mu_0}{4\pi} \int \frac{\nabla^1 X \, M}{R} dv^1 - \frac{\mu_0}{4\pi} \int \nabla^1 X \frac{M}{R} dv^1$$

Applying the Vector Identity

$$\int_{v^1} \nabla_1 X \, F \, dv^1 = -\int_{S^1} F \, X \, dS$$

$$A = \frac{0}{4}\int_{v^1} \frac{\nabla^1 X M}{R}\, dv^1 + \frac{0}{4}\oint_{S^1} \frac{M \; X a_n}{R}\, dS^1$$

$$A = \frac{0}{4}\int_{v^1} \frac{\nabla^1 R X M}{R}\, dv^1 + \frac{0}{4}\oint_{S^1} \frac{M R X a_n}{R}\, dS^1$$

$$A = \frac{0}{4}\int_{v^1} \frac{J_b}{R}\, dv^1 + \frac{0}{4}\oint_{S^1} \frac{K_b}{R}\, dS^1$$

Comparing the equations

$$J_b = \nabla \; X M$$

$$K_b = M \; X a_n$$

J_b is the bound volume current density or magnetizing volume current density.

K_b is the bound surface current density.

In free Space

$$\nabla X H = J_f \quad \text{Or} \quad \nabla X \left(\frac{B}{\mu_0} \right) = J_f$$

J_f is the free current volume density

In a medium where M is not equal to zero, then

$$\nabla X \left(\frac{B}{\mu_0} \right) = J_f + J_b = J = \nabla X B \; \nabla X M$$

or

$$B = \mu_0 \left(H + M \right)$$

For linear materials, M depends linearly on H such that

$$M = \chi_m H$$

χ_m is called Magnetic susceptibility of the medium.

$$B = \mu_0 (1 + \chi_m) H$$

$$B = \mu_0 \mu_r H$$

Where

$$\mu_r = 1 + \chi_m = \frac{\mu}{\mu_0}$$

μ is called as the permeability of the material

μ_r is called as the relative permeability of the material

Scalar and Vector Magnetic

- Potentials
- Vector Potentials due to simple
- configurations Self and Mutual
- Inductances
- Determination of
- inductance Energy

Magnetic Potential

- we can define a potential associated with magnetostatic field B.
- The magnetic potential could be scalar or vector

Scalar Magnetic Potential

- We define the *magnetic scalar potential Vm*.

$$\mathbf{H} = -\nabla V_m$$

$$\mathbf{J} = \nabla \times \mathbf{H} = \nabla \times (-\nabla V_m) = 0$$

Thus the magnetic scalar potential V_m is only defined in a region where J = 0

Vector Magnetic Potential

$$\mathbf{A} = \int_L \frac{\mu_o I \, d\mathbf{l}}{4\pi R}$$ for line current

$$\mathbf{A} = \int_S \frac{\mu_o \mathbf{K} \, dS}{4\pi R}$$ for surface current

$$\mathbf{A} = \int_v \frac{\mu_o \mathbf{J}\, dv}{4\pi R}$$

for volume current

$$\mathbf{B} = \nabla \times \mathbf{A}$$

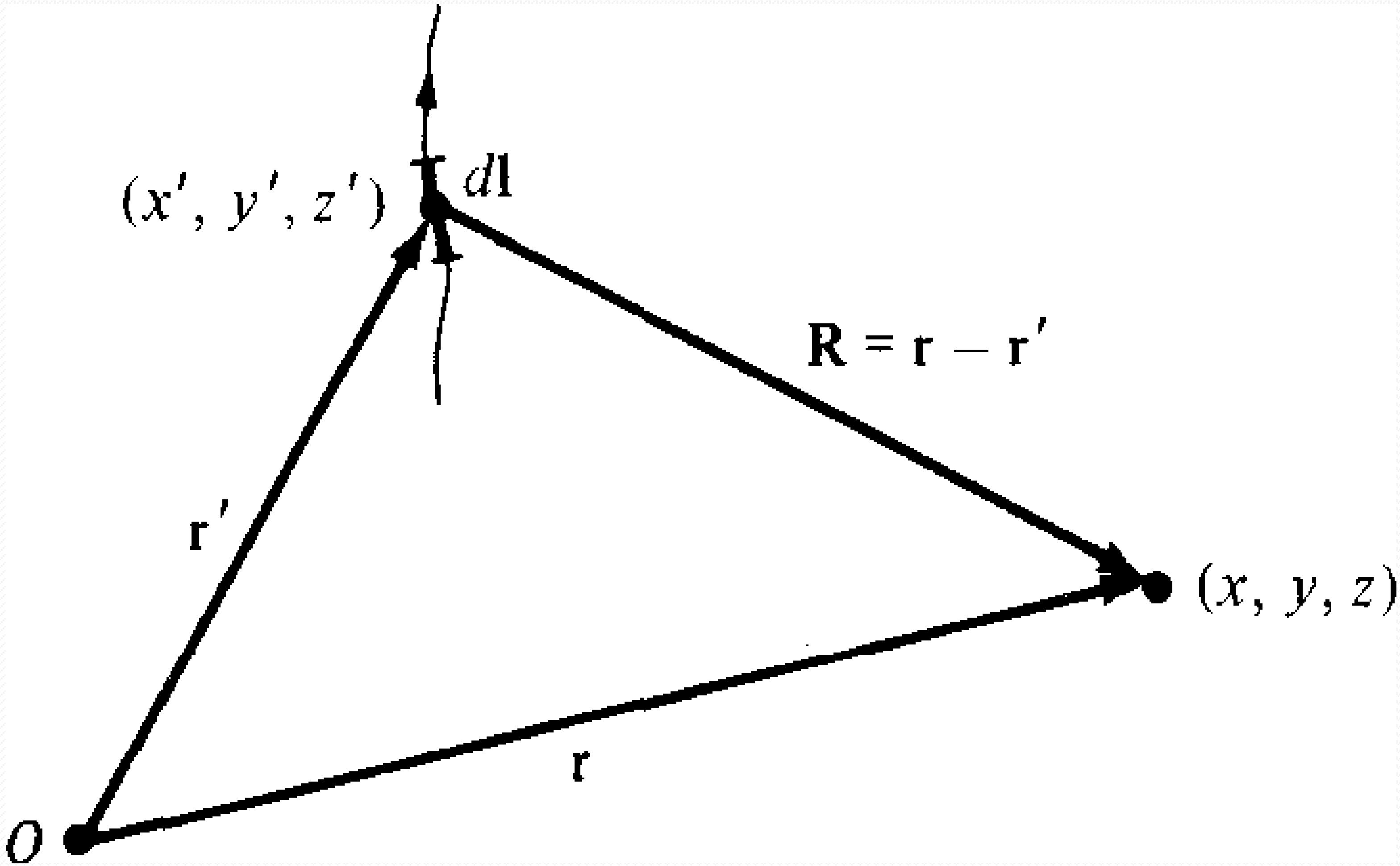
Proof

- We Know that

$$\mathbf{B} = \frac{\mu_o}{4\pi} \int_L \frac{I\, d\mathbf{l}' \times \mathbf{R}}{R^3}$$

$$R = |\mathbf{r} - \mathbf{r}'| = [(x - x')^2 + (y - y')^2 + (z - z')^2]^{1/2}$$

$$\nabla\left(\frac{1}{R}\right) = -\frac{(x - x')\mathbf{a}_x + (y - y')\mathbf{a}_y + (z - z')\mathbf{a}_z}{[(x - x')^2 + (y - y')^2 + (z - z')^2]^{3/2}} = -\frac{\mathbf{R}}{R^3}$$



$d\mathbf{l}$
(x', y', z')
$\mathbf{R} = \mathbf{r} - \mathbf{r}'$
$\mathbf{r}'$
(x, y, z)
$\mathbf{r}$
O

$$\frac{\mathbf{R}}{R^3} = -\nabla\left(\frac{1}{R}\right) \qquad \left(= \frac{\mathbf{a}_R}{R^2}\right)$$

$$\mathbf{B} = -\frac{\mu_o}{4\pi}\int_L I\, d\mathbf{l}' \times \nabla\left(\frac{1}{R}\right)$$

$$\nabla \times (f\mathbf{F}) = f\nabla \times \mathbf{F} + (\nabla f) \times \mathbf{F}$$

$$d\mathbf{l}' \times \nabla\left(\frac{1}{R}\right) = \frac{1}{R}\nabla \times d\mathbf{l}' - \nabla \times \left(\frac{d\mathbf{l}'}{R}\right)$$

$$d\mathbf{l}' \times \nabla\left(\frac{1}{R}\right) = -\nabla \times \frac{d\mathbf{l}'}{R}$$

$$\mathbf{A} = \int_L \frac{\mu_o I \, d\mathbf{l}'}{4\pi R}$$

$$\Psi = \int_S \mathbf{B} \cdot d\mathbf{S} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \oint_L \mathbf{A} \cdot d\mathbf{l}$$

$$\Psi = \oint_L \mathbf{A} \cdot d\mathbf{l}$$

We Know that

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

$$\mathbf{B} = \nabla \times \oint_L \frac{\mu_0 I \, d\mathbf{l}'}{4\pi R} = \frac{\mu_0 I}{4\pi} \oint_L \nabla \times \frac{1}{R} \, d\mathbf{l}',$$

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \oint_L \left[\frac{1}{R} \nabla \times d\mathbf{l}' + \left(\nabla \frac{1}{R} \right) \times d\mathbf{l}' \right]$$

$$\frac{1}{R} = [(x - x')^2 + (y - y')^2 + (z - z')^2]^{-1/2}$$

$$\nabla\left[\frac{1}{R}\right] = -\frac{(x - x')\mathbf{a}_x + (y - y')\mathbf{a}_y + (z - z')\mathbf{a}_z}{[(x - x')^2 + (y - y')^2 + (z - z')^2]^{3/2}} = -\frac{\mathbf{a}_R}{R^2}$$

$$\mathbf{B} = \frac{\mu_o I}{4\pi} \oint_L \frac{d\mathbf{l} \times \mathbf{a}_R}{R^2}$$

$$\nabla \times \mathbf{B} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

$$\boxed{\nabla \cdot \mathbf{A} = 0}$$

$$\nabla^2 \mathbf{A} = -\mu_o \nabla \times \mathbf{H}$$

$$\boxed{\nabla^2 \mathbf{A} = -\mu_o \mathbf{J}}$$

This equation is called vector Poisson's equation. In Cartesian form these can be written as

$$\nabla^2 A_x = -\mu_o J_x$$

$$\nabla^2 A_y = -\mu_o J_y$$

$$\nabla^2 A_z = -\mu_o J_z$$

$$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S \nabla \times \mathbf{H} \cdot d\mathbf{S}$$

$$= \frac{1}{\mu_o} \int_S \nabla \times (\nabla \times \mathbf{A}) \cdot d\mathbf{S}$$

$$\nabla \times \nabla \times \mathbf{A} = -\nabla^2 \mathbf{A} = \mu_o \mathbf{J}$$

$$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} = I$$

Flux Linkages

- A circuit (or closed conducting path) carrying current /produces a magnetic field B which causes a flux $\psi = \int B \bullet dS$ to pass through each turn of the circuit as shown in Figure.
- If the circuit has N identical turns, we define *the flux linkage* λ as

$$\lambda = N\psi$$

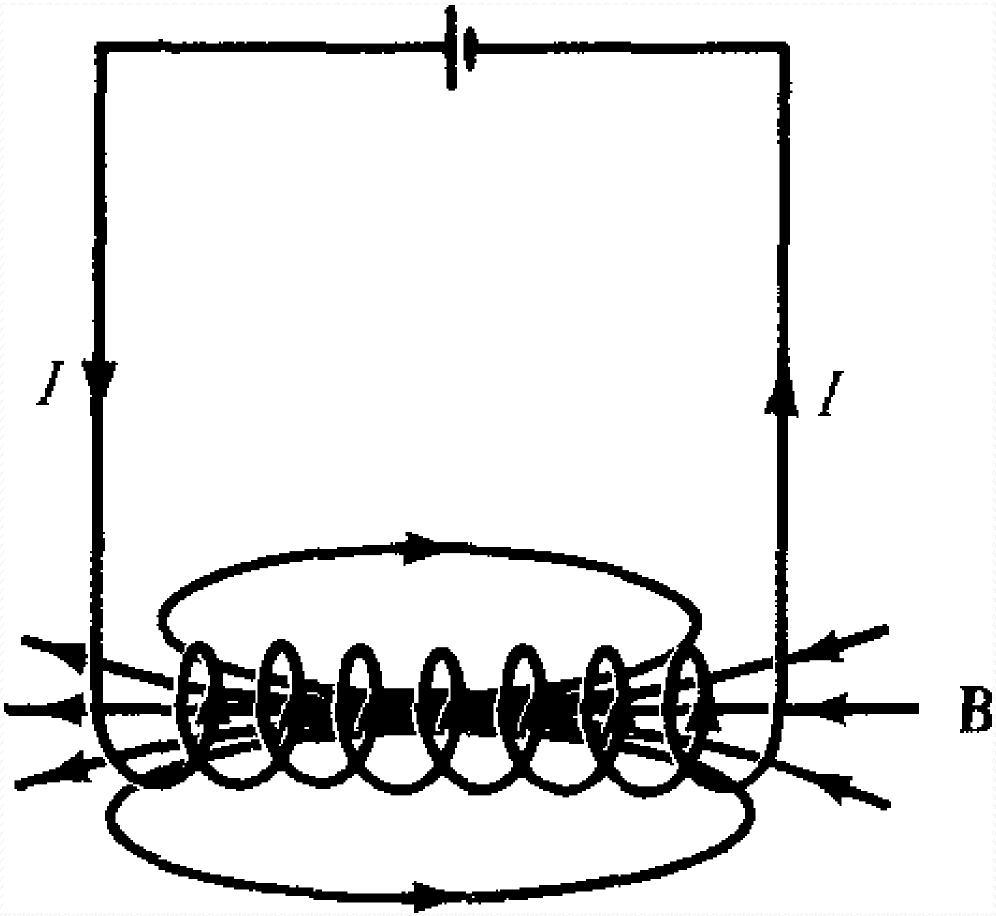
- If the medium surrounding the circuit is linear, the flux linkage Xis proportional to the current I producing it.

Inductance

Inductance L of an inductor as the ratio of the magnetic flux linkage λ to the current I through the inductor

$$L = \frac{\lambda}{I} = \frac{N\Psi}{I}$$

The unit of inductance is the henry (H) which is the same as webers/ampere.



Inductance is a measure of how much magnetic energy is stored in an inductor.

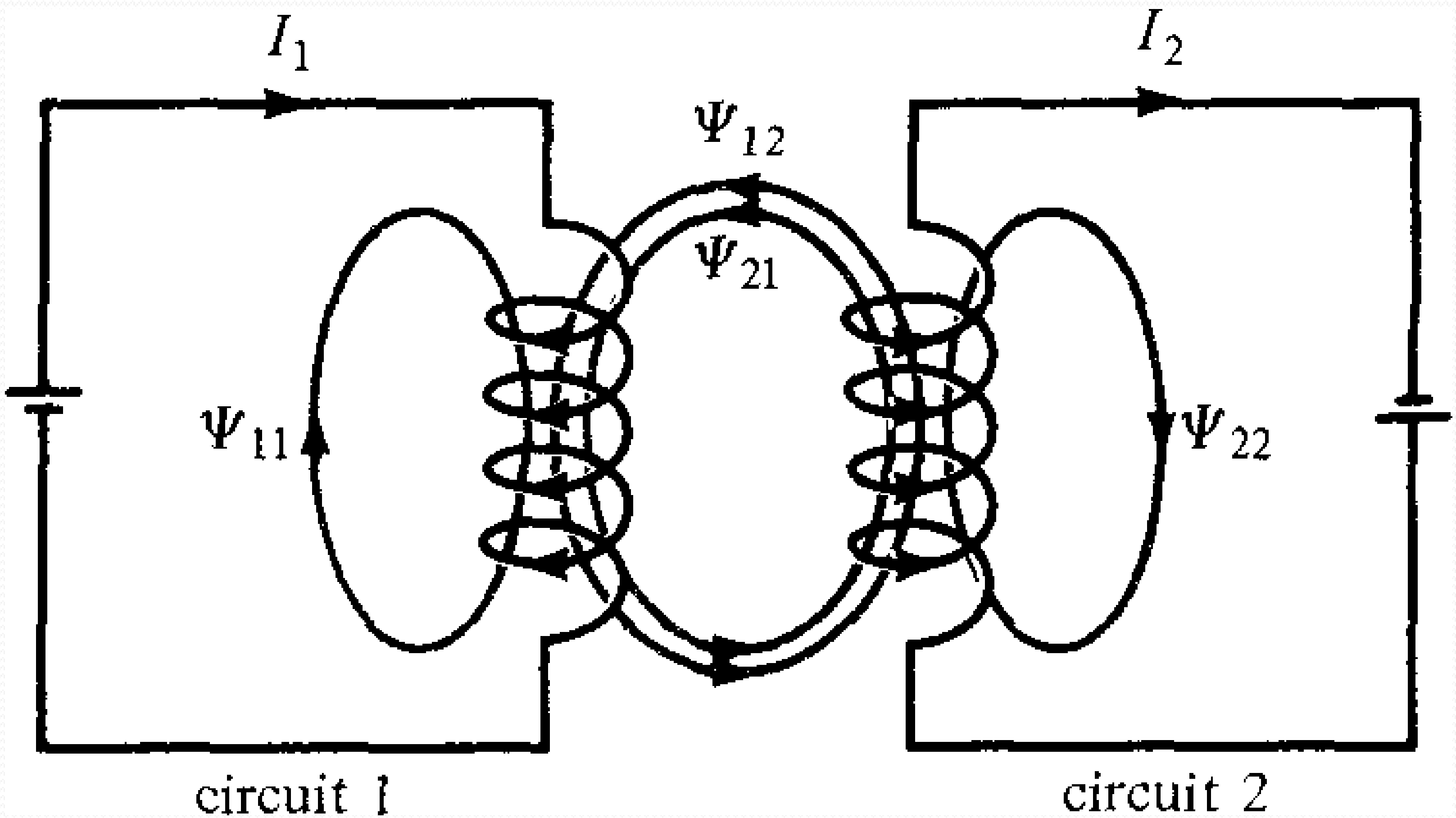
The magnetic energy (in joules) stored in an inductor is expressed as

$$W_m = \frac{1}{2} L I^2$$

$$\boxed{L = \frac{2 W_m}{I^2}}$$

Mutual Inductance

- If instead of having a single circuit we have two circuits carrying current I_1 and I_2 as shown in Figure, a magnetic interaction exists between the circuits.
- Four component fluxes ψ_{11}, ψ_{12}, ψ_{21} and ψ_{22} are produced.
- The flux ψ_{12}, for example, is the flux passing through circuit 1 due to current I_2 in circuit 2. If B_2 in the field due to I_2 and S_1 is the area of circuit 1, then



I_1
I_2
Ψ_{12}
Ψ_{21}
Ψ_{11}
Ψ_{22}
circuit 1
circuit 2

$$\Psi_{12} = \int_{S_1} \mathbf{B}_2 \cdot d\mathbf{S}$$

We define the *mutual inductance* M_{12} as the ratio of the flux linkage $\lambda_{12} = N_1 \psi_{12}$ on circuit 1 to current I_2, that is

$$M_{21} = \frac{\lambda_{21}}{I_1} = \frac{N_2 \Psi_{21}}{I_1}$$

$$M_{12} = M_{21}$$

We define the self-inductance of circuits 1 and 2, respectively, as

$$L_1 = \frac{\lambda_{11}}{I_1} = \frac{N_1 \Psi_1}{I_1}$$

$$L_2 = \frac{\lambda_{22}}{I_2} = \frac{N_2 \Psi_2}{I_2}$$

$$\begin{aligned} W_m &= W_1 + W_2 + W_{12} \\ &= \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 \pm M_{12} I_1 I_2 \end{aligned}$$

Magnetic Energy

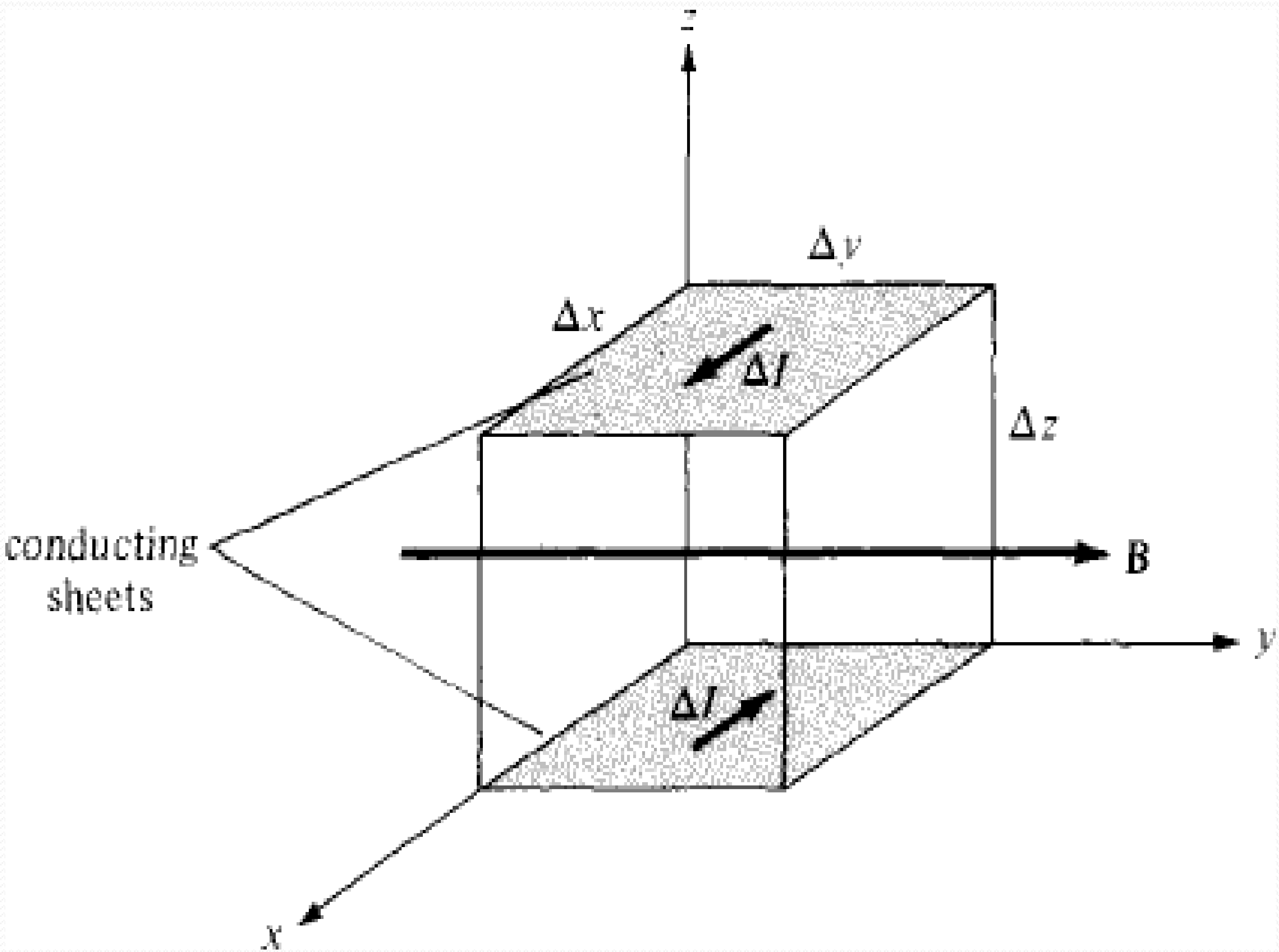
Just as the potential energy in an electrostatic field was derived as

$$W_E = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\,dv = \frac{1}{2}\int \varepsilon E^2\,dv$$

$$\Delta L = \frac{\Delta\Psi}{\Delta I} = \frac{\mu H\,\Delta x\,\Delta z}{\Delta I}$$

$$\Delta W_m = \frac{1}{2}\Delta L\,\Delta I^2 = \frac{1}{2}\mu H^2\,\Delta x\,\Delta y\,\Delta z$$

$$\Delta W_m = \frac{1}{2}\mu H^2\,\Delta v$$


z
Δy
Δx
ΔI
Δz
conducting
sheets
B
y
ΔI
x

$$w_m = \lim_{\Delta v \to 0} \frac{\Delta W_m}{\Delta v} = \frac{1}{2}\mu H^2$$

$$w_m = \frac{1}{2}\mu H^2 = \frac{1}{2} B \cdot H = \frac{B^2}{2\mu}$$

$$W_m = \int w_m \, dv$$

$$\boxed{W_m = \frac{1}{2}\int \mathbf{B} \cdot \mathbf{H} \, dv = \frac{1}{2}\int \mu H^2 \, dv}$$

Inductance of a Solenoid

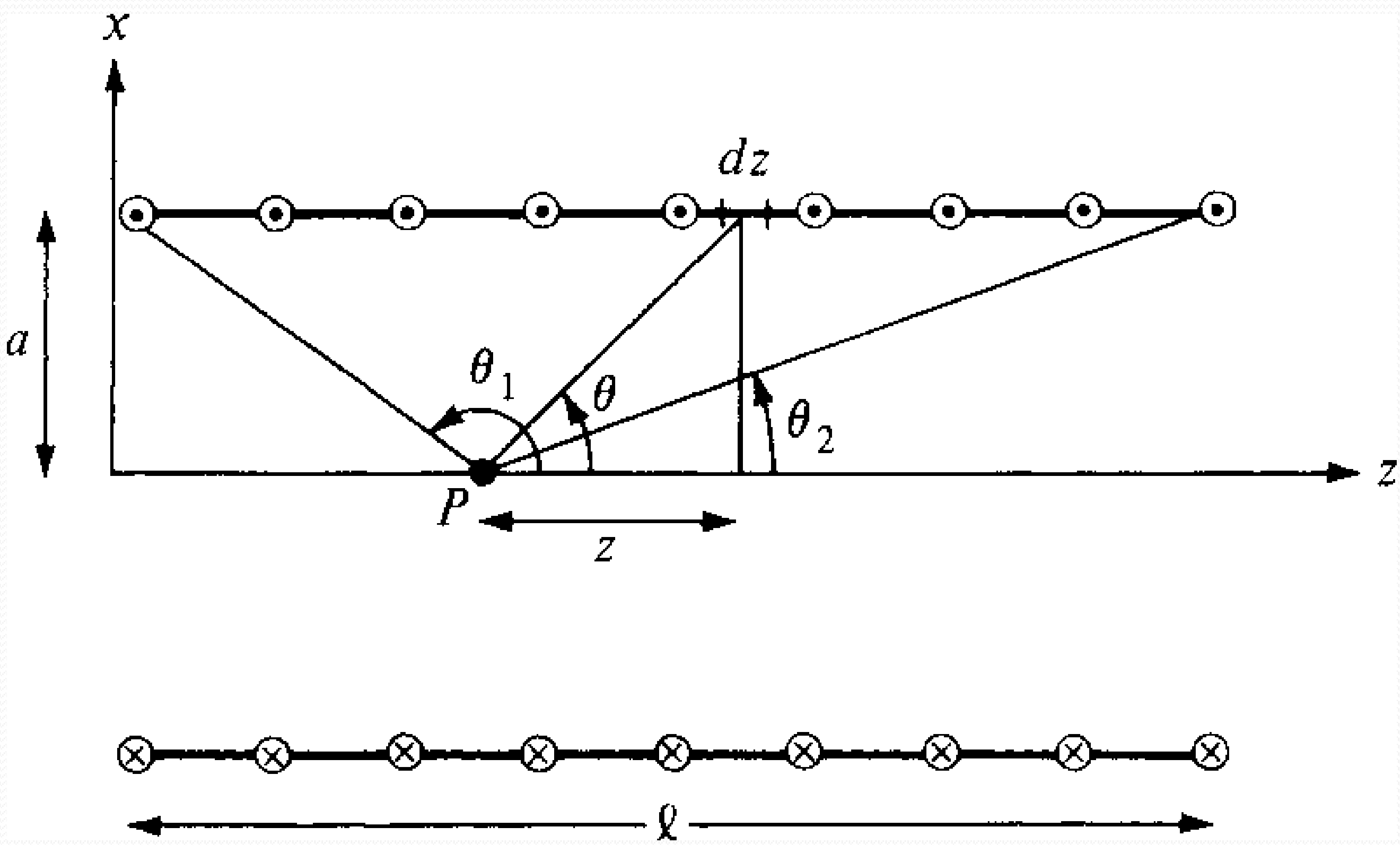
For an infinitely long solenoid, the magnetic flux inside the solenoid per unit length is

$$B = \mu H = \mu I n$$

$$\Psi = BS = \mu I n S$$

$$\lambda' = \frac{\lambda}{\ell} = n\Psi = \mu n^2 I S$$

$$L' = \frac{L}{\ell} = \frac{\lambda'}{I} = \mu n^2 S$$



x
dz
a
θ1
θ
θ2
z
P
z
ℓ

For length l of the cable,

$$\lambda_1 = \int_{\rho=0}^{a} \int_{z=0}^{\ell} \frac{\mu I \rho^3 \, d\rho \, dz}{2\pi a^4} = \frac{\mu I \ell}{8\pi}$$

$$L_{in} = \frac{\lambda_1}{I} = \frac{\mu \ell}{8\pi}$$

The internal inductance per unit length, given by

$$\boxed{L'_{in} = \frac{L_{in}}{\ell} = \frac{\mu}{8\pi}} \qquad \text{H/m}$$

flux linkages between the inner and the outer conductor as in Figure. For a differential shell of thickness dρ.

$$d\Psi_2 = B_2\, d\rho\, dz = \frac{\mu I}{2\pi\rho}\, d\rho\, dz$$

$$\lambda_2 = \Psi_2 = \int_{\rho=a}^{b} \int_{z=0}^{\ell} \frac{\mu I\, d\rho\, dz}{2\pi\rho} = \frac{\mu I \ell}{2\pi} \ln \frac{b}{a}$$

$$L_{\text{ext}} = \frac{\lambda_2}{I} = \frac{\mu \ell}{2\pi} \ln \frac{b}{a}$$

$$L = L_{\text{in}} + L_{\text{ext}} = \frac{\mu \ell}{2\pi}\left[\frac{1}{4} + \ln\frac{b}{a}\right]$$

The inductance per length is

$$\boxed{L' = \frac{L}{\ell} = \frac{\mu}{2\pi}\left[\frac{1}{4} + \ln\frac{b}{a}\right]} \qquad \text{H/m}$$

Unit IV
TIME VARYING FIELDS AND MAXWELL’S EQUATIONS

Displacement Current

Ampere's Law – Curl H Equation

(quasi) Static field

$$\nabla \times \vec{H} = \vec{j}$$

Time varying field

$$\nabla \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}$$

Displacement current density

Integral Form of Ampere's Law for time varying fields

$$\oint_C \vec{H} \bullet d\vec{l} = I_c + \int_S \frac{\partial \vec{D}}{\partial t} \bullet d\vec{s}$$

Displacement current

I_C – Conduction Current [A] linked to a conductivity property

$\vec{D}$ – Electric Flux Density (Electric Displacement) [in C/unit area]

$\vec{j}_c$ – Conduction Current Density (in A/unit area)

Displacement Current

$$\oint_C \vec{H} \bullet d\vec{l} = I_c + I_d = I$$ → *Total current*

Conduction current density

$$I_c = \int \vec{j}_c \bullet d\vec{s} = \int \sigma\vec{E} \bullet d\vec{s} \qquad (\vec{j}_c = \sigma\vec{E})$$

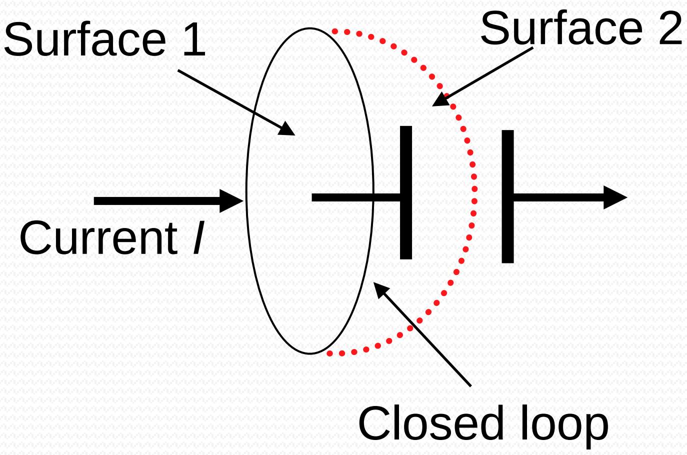
$$I_d = \int_S \vec{j}_d \bullet d\vec{s} = \int_S \frac{\partial \vec{D}}{\partial t} \bullet d\vec{s}$$

Displacement current density

Connection between electric and magnetic fields under time varying conditions

Need for Displacement Current

- Faraday: vary B-field, generate E-field
- Maxwell: varying E-field should then produce a B-field, but not covered by Ampère's Law.



- Apply Ampère to surface 1 (flat disk): line integral of $\boldsymbol{B} = \mu_0 I$
- Applied to surface 2, line integral is zero since no current penetrates the deformed surface.
- In capacitor,

$$E = \frac{Q}{\varepsilon_0 A}$$

so

$$I = \frac{dQ}{dt} = \varepsilon_0 A \frac{dE}{dt}$$

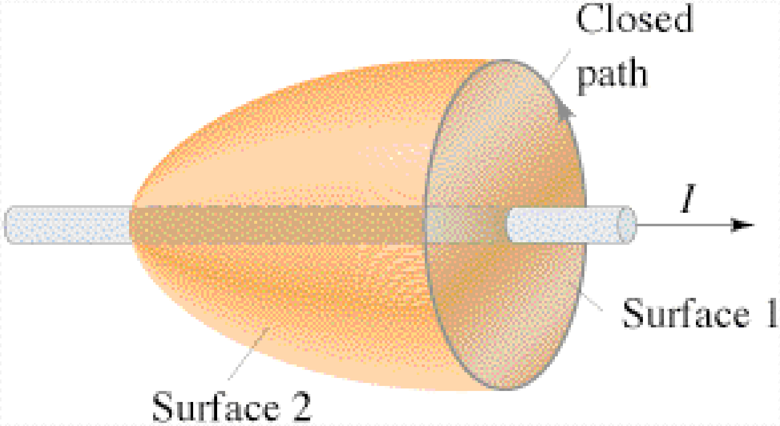
- Displacement current density is

$$\vec{j}_d = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \wedge \vec{B} = \mu_0 \left(\vec{j} + \vec{j}_d \right) = \mu_0 \vec{j} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Maxwell's Equations

- Start with Ampere's Law $\sum B_{\parallel} \Delta l = \mu_0 I$



Earlier, we just went on a closed path enclosing surface 1. But according to Ampere's Law, we could have considered surface 2. The current enclosed is the same as for surface 1. We can say that the current flowing into any volume must equal that coming out.

Maxwell's Equations

- While the capacitor is discharging, a current flows
- The electric field between the plates of the capacitor is decreasing as current flows
- Maxwell said the changing electric field is equivalent to a current
- He called it the **displacement current**

Maxwell's Equations

Name	Partial differential form	Integral form
Gauss's law:	$\nabla \cdot \mathbf{D} = \rho$	$\oint_A \mathbf{D} \cdot d\mathbf{A} = Q_{encl}$
Gauss's law for magnetism:	$\nabla \cdot \mathbf{B} = 0$	$\oint_A \mathbf{B} \cdot d\mathbf{A} = 0$
Faraday's law of induction:	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_S \mathbf{E} \cdot ds = -\frac{d\Phi_B}{dt}$
Ampere's law + Maxwell's extension:	$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$	$\oint_S \mathbf{H} \cdot ds = I_{\text{enc}} + \frac{d\mathbf{\Phi_D}}{dt}$

Maxwell's Equations

Relate Electric and Magnetic fields generated by charge and current distributions.

E = electric field

D = electric displacement

H = magnetic field

B = magnetic flux density

ρ= charge density

j = current density

μ_0 (permeability of free space) = 4π 10^{-7}

ε_0 (permittivity of free space) = 8.854 10^{-12}

c (speed of light) = 2.99792458 10^{8} m/s

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \wedge \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}$$

In vacuum $\vec{D} = \varepsilon_0 \vec{E}, \quad \vec{B} = \mu_0 \vec{H}, \quad \varepsilon_0 \mu_0 c^2 = 1$

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

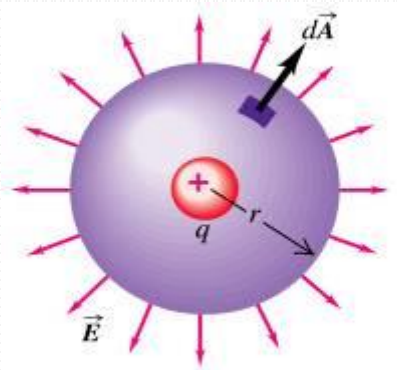
Maxwell's 1st Equation

Equivalent to Gauss' Flux Theorem:

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \quad \Leftrightarrow \quad \iiint_V \nabla \cdot \vec{E}\, dV = \oiint_S \vec{E} \cdot d\vec{S} = \frac{1}{\varepsilon_0} \iiint_V \rho\, dV = \frac{Q}{\varepsilon_0}$$

The flux of electric field out of a closed region is proportional to the total electric charge Q enclosed within the surface.

A point charge *q* generates an electric field



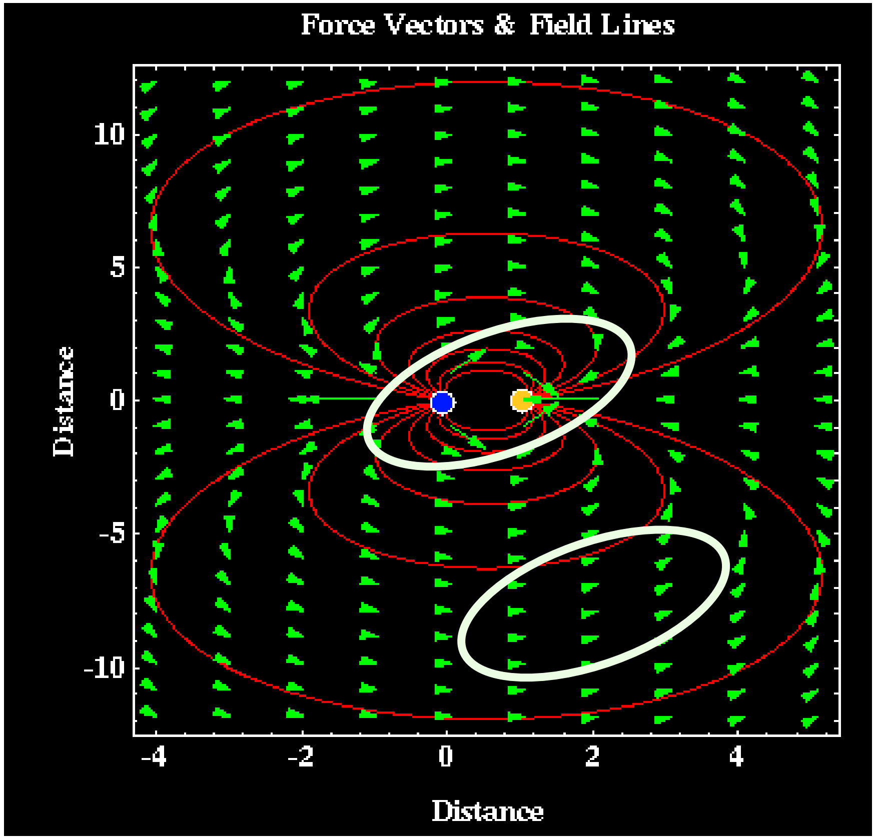
$$\vec{E} = \frac{q}{4\pi\varepsilon_0 r^3} \vec{r}$$

$$\iint_{sphere} \vec{E} \cdot d\vec{S} = \frac{q}{4\pi\varepsilon_0} \iint_{sphere} \frac{dS}{r^2} = \frac{q}{\varepsilon_0}$$

Area integral gives a measure of the net charge enclosed; divergence of the electric field gives the density of the sources.

$$\nabla \cdot \vec{B} = 0$$

Maxwell's 2nd Equation



Gauss' law for magnetism:

$$\nabla \cdot \vec{B} = 0 \quad \Leftrightarrow \quad \oiint \vec{B} \cdot d\vec{S} = 0$$

The net magnetic flux out of any closed surface is zero. Surround a magnetic dipole with a closed surface. The magnetic flux directed inward towards the south pole will equal the flux outward from the north pole.

If there were a magnetic monopole source, this would give a non-zero integral.

Gauss' law for magnetism is then a statement that ***There are no magnetic monopoles***

$$\nabla \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Maxwell's 3rd Equation

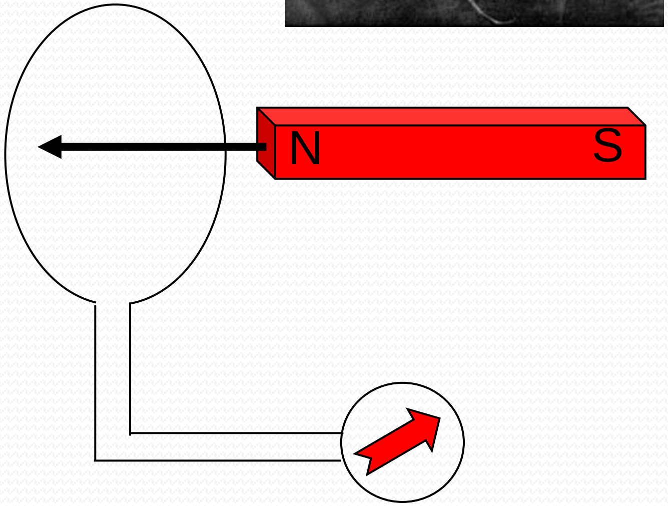
Equivalent to Faraday's Law of Induction:

$$\iint_S \nabla \wedge \vec{E} \cdot d\vec{S} = -\iint_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$$

$$\Leftrightarrow \quad \oint_C E \cdot d\vec{l} = -\frac{d}{dt}\iint_S \vec{B} \cdot d\vec{S} = -\frac{d\Phi}{dt}$$

(for a fixed circuit *C*)

The electromotive force round a circuit $\varepsilon = \oint \vec{E} \cdot d\vec{l}$ is proportional to the rate of change of flux of magnetic field, $\Phi = \iint \vec{B} \cdot d\vec{S}$ through the circuit.



Faraday's Law is the basis for electric generators. It also forms the basis for inductors and transformers.

Potentials

- Magnetic vector potential:

$$\nabla \cdot \vec{B} = 0 \quad \Leftrightarrow \quad \exists\, \vec{A} \text{ such that } \vec{B} = \nabla \wedge \vec{A}$$

- Electric scalar potential:

$$\nabla \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t}\left(\nabla \wedge \vec{A}\right) = -\nabla \wedge \frac{\partial \vec{A}}{\partial t} \quad \Leftrightarrow \quad \nabla \wedge \left(E + \frac{\partial \vec{A}}{\partial t} \right) = 0$$

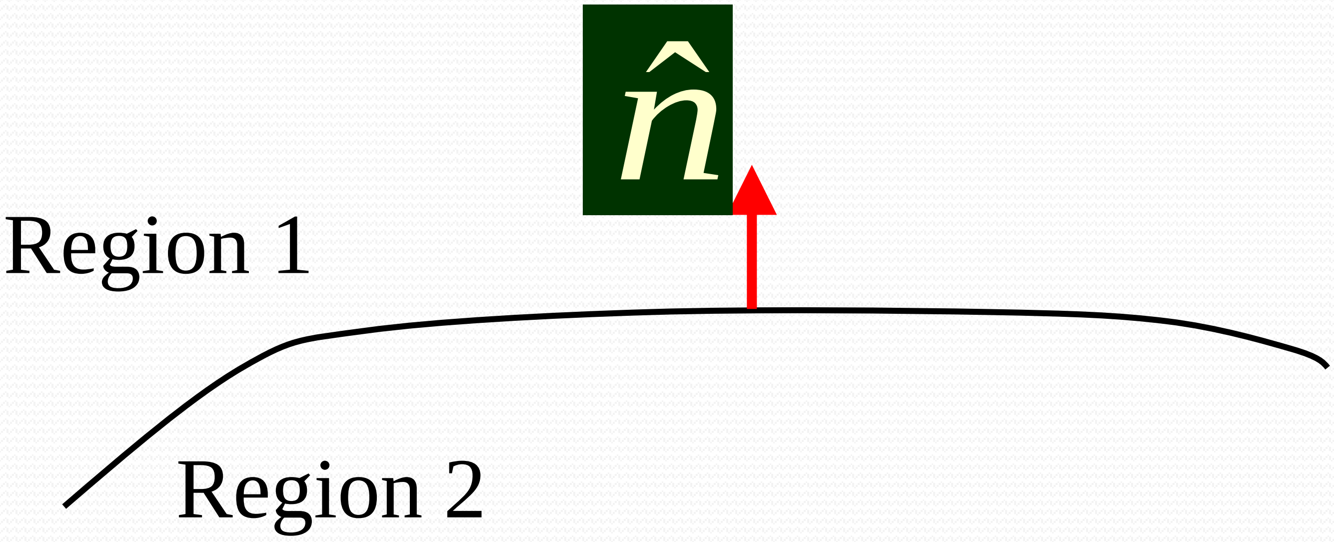
$$\Leftrightarrow \quad \exists\, \phi \text{ with } \vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla \phi, \quad \text{so} \quad \vec{E} = -\nabla \phi - \frac{\partial \vec{A}}{\partial t}$$

- Lorentz Gauge: $\phi \to \phi + f(t), \quad \vec{A} \to \vec{A} + \nabla \chi$

Use freedom to set

$$\frac{1}{c^2}\frac{\partial \phi}{\partial t} + \nabla \cdot \vec{A} = 0$$

Electromagnetic Boundary Conditions



Electromagnetic Boundary Conditions

$$\hat{n} \times \left(\underline{E}_1 - \underline{E}_2 \right) = -\underline{K}_S$$

$$\hat{n} \times \left(\underline{H}_1 - \underline{H}_2 \right) = \underline{J}_S$$

$$\hat{n} \cdot \left(\underline{D}_1 - \underline{D}_2 \right) = q_{es}$$

$$\hat{n} \cdot \left(\underline{B}_1 - \underline{B}_2 \right) = q_{ms}$$

Phasor Representation of a Time-Harmonic Field

- A ***phasor*** is a complex number representing the amplitude and phase of a sinusoid of known frequency.

phasor

$$A\cos(\omega t+\theta) \Leftrightarrow Ae^{j\theta}$$

time domain

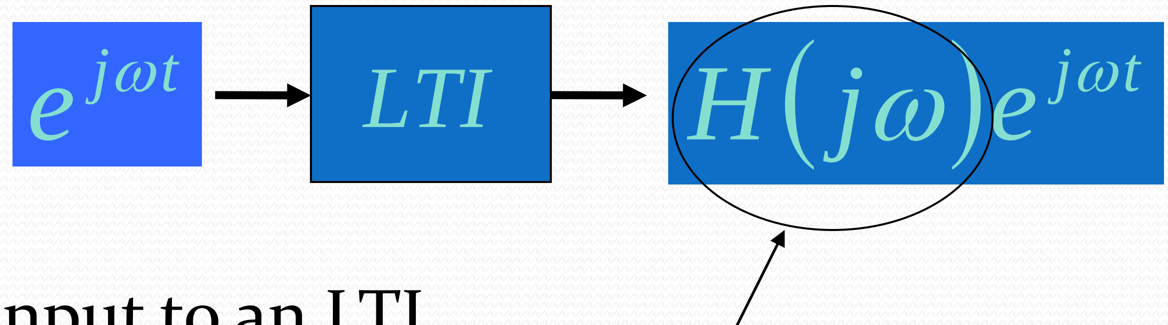
frequency domain

Phasor Representation of a Time-Harmonic Field

- ***Phasors*** are an extremely important concept in the study of classical electromagnetics, circuit theory, and communications systems.
- Maxwell's equations in simple media, circuits comprising linear devices, and many components of communications systems can all be represented as ***linear time-invariant*** (***LTI***) systems. (Formal definition of these later in the course …)
- The eigenfunctions of any LTI system are the complex exponentials of the form:

$$e^{j\omega t}$$

Phasor Representation of a Time-Harmonic Field



- If the input to an LTI system is a sinusoid of frequency ω, then the output is also a sinusoid of frequency ω (with different amplitude and phase).

A complex constant (for fixed ω); as a function of ω gives the frequency response of the LTI system.

Phasor Representation of a Time-Harmonic Field

- The amplitude and phase of a sinusoidal function can also depend on position, and the sinusoid can also be a vector function:

$$\hat{a}_A A(\underline{r}) \cos\left(\omega t - \theta(\underline{r})\right) \Leftrightarrow \hat{a}_A A(\underline{r})\, e^{j\theta(\underline{r})}$$

Phasor Representation of a Time-Harmonic Field

- Given the phasor (frequency-domain) representation of a time-harmonic vector field, the time-domain representation of the vector field is obtained using the recipe:

$$\underline{E}(\underline{r},t) = \mathrm{Re}\left\{\underline{E}(\underline{r})\, e^{j\omega t}\right\}$$

Phasor Representation of a Time-Harmonic Field

- ***Phasors*** can be used provided all of the media in the problem are ***linear*** $\Rightarrow$ no frequency conversion.
- When phasors are used, integro-differential operators in time become algebraic operations in frequency, e.g.:

$$\frac{\partial \underline{E}(\underline{r},t)}{\partial t} \Leftrightarrow j\omega \underline{E}(\underline{r})$$

Time-Harmonic Maxwell's Equations

- If the sources are time-harmonic (sinusoidal), and all media are linear, then the electromagnetic fields are sinusoids of the same frequency as the sources.
- In this case, we can simplify matters by using Maxwell's equations in the *frequency-domain*.
- Maxwell's equations in the frequency-domain are relationships between the phasor representations of the fields.

Maxwell's Equations in Differential Form for Time-Harmonic Fields

$$\nabla \times \underline{E} = -j\omega \underline{B} - \underline{K}_c - \underline{K}_i$$

$$\nabla \times \underline{H} = j\omega \underline{D} + \underline{J}_c + \underline{J}_i$$

$$\nabla \cdot \underline{D} = q_{ev}$$

$$\nabla \cdot \underline{B} = q_{mv}$$

Maxwell's Equations in Differential Form for Time-Harmonic Fields in Simple Medium

$$\nabla \times \underline{E} = -\left(j\omega\mu + \sigma_m\right)\underline{H} - \underline{K}_i$$

$$\nabla \times \underline{H} = \left(j\omega\varepsilon + \sigma\right)\underline{E} + \underline{J}_i$$

$$\nabla \cdot \underline{E} = \frac{q_{ev}}{\varepsilon}$$

$$\nabla \cdot \underline{H} = \frac{q_{mv}}{\mu}$$

Unit V
PLANE ELECTROMAGNETIC WAVES

Plane EM Wave in a Lossy and Lossless Media:

$$\nabla \times H = \vec{J} + j\omega\varepsilon\vec{E} = \sigma\vec{E} + j\omega\varepsilon\vec{E} = j\omega(\varepsilon - j\frac{\sigma}{\omega})\vec{E} = j\omega\varepsilon_c\vec{E}\ ,\ \varepsilon_c = \varepsilon - j\frac{\sigma}{\omega} = \varepsilon' - j\varepsilon''.$$

Similarly, $\mu_c = \mu' - j\mu''$

Complex wave number: $k_c = \omega\sqrt{\mu\varepsilon_c}$. Loss tangent: $\tan\delta_c \approx \varepsilon''/\varepsilon' = \frac{\sigma}{\omega\varepsilon}$

Propagation constant: $\gamma = jk_c = j\omega\sqrt{\mu\varepsilon_c} = \alpha + j\beta = j\omega\sqrt{\mu\varepsilon}(1+\frac{\sigma}{j\omega\varepsilon})^{1/2}$

$$E \propto e^{-\gamma z} = e^{-jk_c z} = e^{-\alpha z}\cdot e^{-j\beta z}$$

If the medium is lossless, α=0; else if the medium is lossy, α>0.

Phase constant: $\beta = \frac{2\pi}{\lambda}$

$$\Rightarrow \alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}[\sqrt{1+(\frac{\sigma}{\omega\varepsilon})^2} - 1]}\ ,\ \beta = \omega\sqrt{\frac{\mu\varepsilon}{2}[\sqrt{1+(\frac{\sigma}{\omega\varepsilon})^2} + 1]}$$

Case 1 Low-loss Dielectric: $\frac{\sigma}{\omega\varepsilon} << 1 \Rightarrow \alpha \approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$, $\beta \approx \omega\sqrt{\mu\varepsilon}[(1+\frac{1}{8}(\frac{\sigma}{\omega\varepsilon})^2]$

Intrinsic impedance: $\eta_c \approx \sqrt{\frac{\mu}{\varepsilon}}(1+j\frac{\sigma}{2\omega\varepsilon})$

Phase velocity: $v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon_c}} \approx \frac{1}{\sqrt{\mu\varepsilon}}[1-\frac{1}{8}(\frac{\sigma}{\omega\varepsilon})^2]$

Case 2 Good Conductor: $\frac{\sigma}{\omega\varepsilon} >> 1 \Rightarrow \alpha = \beta \approx \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f\mu\sigma}$,

and $\eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} \approx (\sqrt{\frac{j\omega\mu}{\sigma}}) = (1+j)\sqrt{\frac{\pi f\mu}{\sigma}} = (1+j)\frac{\alpha}{\sigma}$

Phase velocity: $v_p = \frac{\omega}{\beta} \approx \sqrt{\frac{2\omega}{\mu\sigma}}$

Skin Depth (depth of penetration): $\delta = \frac{1}{\alpha} = \frac{1}{\sqrt{\pi f\mu\sigma}}$.

For a good conductor, $\delta = \frac{1}{\alpha} \approx \frac{1}{\beta} = \frac{\lambda}{2\pi}$

Electromagnetic waves

- Maxwell's equations predict the existence of electromagnetic waves, later discovered by Hertz.
- No charges, no currents:

$$\nabla \wedge \vec{H} = \frac{\partial \vec{D}}{\partial t} \qquad \nabla \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{D} = 0 \qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \wedge \left(\nabla \wedge \vec{E}\right) = -\nabla \wedge \frac{\partial \vec{B}}{\partial t}$$

$$= -\frac{\partial}{\partial t}\left(\nabla \wedge \vec{B}\right)$$

$$= -\mu \frac{\partial^2 \vec{D}}{\partial t^2} = -\mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla \wedge \left(\nabla \wedge \vec{E}\right) = \nabla\left(\nabla \cdot \vec{E}\right) - \nabla^2 \vec{E}$$

$$= -\nabla^2 \vec{E}$$

3D wave equation :

$$\nabla^2 \vec{E} = \frac{\partial^2 \vec{E}}{\partial x^2} + \frac{\partial^2 \vec{E}}{\partial y^2} + \frac{\partial^2 \vec{E}}{\partial z^2} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

Nature of Electromagnetic Waves

- A general plane wave with angular frequency ω travelling in the direction of the wave vector ***k*** has the form

$$\vec{E} = \vec{E}_0 \exp[i(\omega t - \vec{k}\cdot\vec{x})] \quad \vec{B} = \vec{B}_0 \exp[i(\omega t - \vec{k}\cdot\vec{x})]$$

- Phase $\omega t - \vec{k}\cdot\vec{x} = 2\pi \times$ number of waves and so is a Lorentz invariant.
- Apply Maxwell's equations

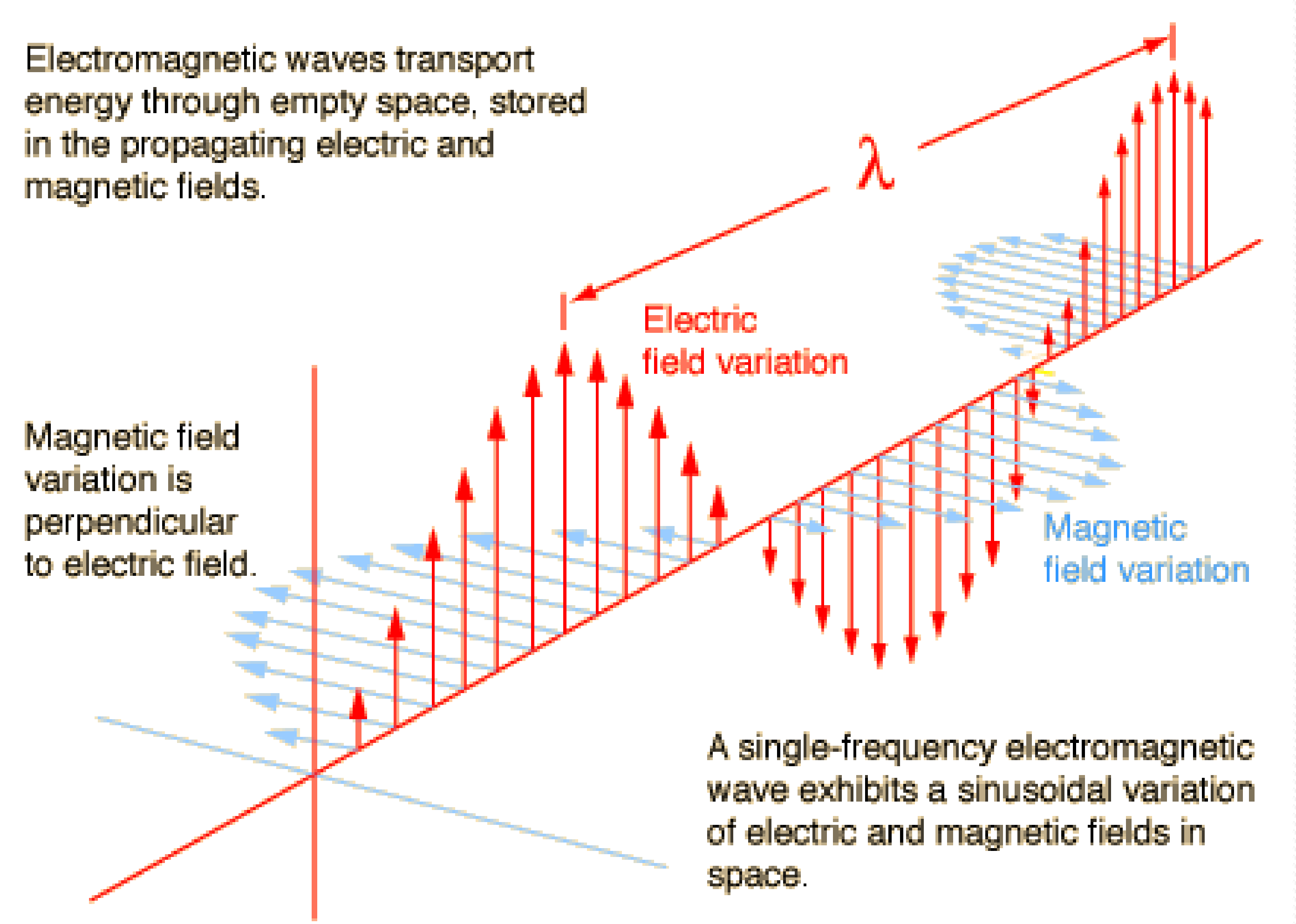
$$\nabla \leftrightarrow -i\vec{k} \qquad \frac{\partial}{\partial t} \leftrightarrow i\omega$$

$$\nabla\cdot\vec{E} = 0 = \nabla\cdot\vec{B} \quad \leftrightarrow \quad \vec{k}\cdot\vec{E} = 0 = \vec{k}\cdot\vec{B}$$

$$\nabla\wedge\vec{E} = -\dot{\vec{B}} \quad \leftrightarrow \quad \vec{k}\wedge\vec{E} = \omega\vec{B}$$

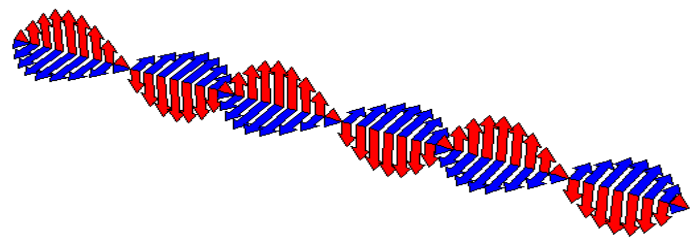
Waves are transverse to the direction of propagation, $\vec{k}$ and $\vec{E}, \vec{B}$ and are mutually perpendicular

Plane Electromagnetic Wave



Plane Electromagnetic Waves

$$\nabla \wedge \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \quad \leftrightarrow \quad \vec{k} \wedge \vec{B} = -\frac{\omega}{c^2}\vec{E}$$



Combined with $\vec{k} \wedge \vec{E} = \omega \vec{B}$

deduce that $\frac{|\vec{E}|}{|\vec{B}|} = \frac{\omega}{k} = \frac{kc^2}{\omega}$

$\Rightarrow$ speed of wave in vacuum is $\frac{\omega}{|\vec{k}|} = c$

Wavelength $\lambda = \frac{2\pi}{|\vec{\mathrm{k}}|}$

Frequency $\nu = \frac{\omega}{2\pi}$

Reminder: The fact that $\omega t - \vec{k} \cdot \vec{x}$ is an invariant tells us that

$$\Lambda = \left(\frac{\omega}{c}, \vec{k} \right)$$

is a Lorentz 4-vector, the 4-Frequency vector. Deduce frequency transforms as

$$\omega' = \gamma\left(\omega - \vec{v} \cdot \vec{k}\right) = \omega \sqrt{\frac{c - v}{c + v}}$$

Waves in a Conducting Medium

$$\vec{E} = \vec{E}_0 \exp[i(\omega t - \vec{k}\cdot\vec{x})] \quad \vec{B} = \vec{B}_0 \exp[i(\omega t - \vec{k}\cdot\vec{x})]$$

- (Ohm's Law) For a medium of conductivity σ, $\vec{j} = \sigma \vec{E}$
- Modified Maxwell: $\nabla \wedge \vec{H} = \vec{j} + \varepsilon \frac{\partial \vec{E}}{\partial t} = \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t}$

$$-i\vec{k} \wedge \vec{H} = \sigma \vec{E} + i\omega\varepsilon \vec{E}$$

(σE: conduction current; iωεE: displacement current)

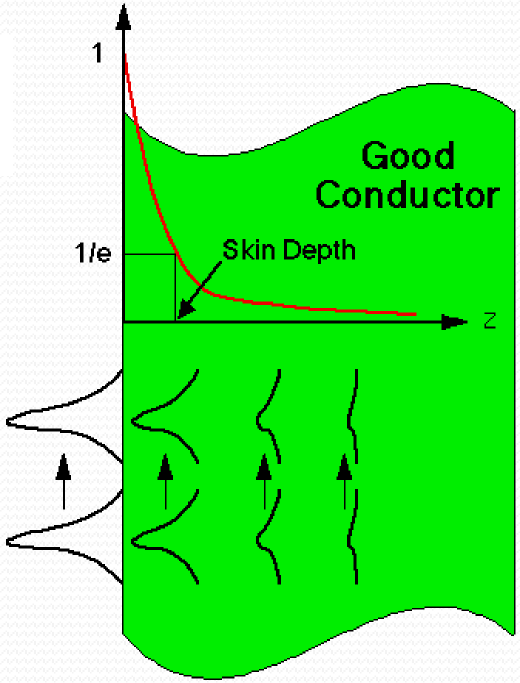
- Put $D = \frac{\sigma}{\omega\varepsilon}$ ← Dissipation factor

$$\text{Copper:} \quad \sigma = 5.8\times 10^{7}, \varepsilon = \varepsilon_0 \quad \Rightarrow \quad D = 10^{12}$$

$$\text{Teflon:} \quad \sigma = 3\times 10^{-8}, \varepsilon = 2.1\varepsilon_0 \quad \Rightarrow \quad D = 2.57\times 10^{-4}$$

Attenuation in a Good Conductor

$$-i\vec{k}\wedge\vec{H}=\sigma\vec{E}+i\omega\varepsilon\vec{E}$$



Combine with $\nabla\wedge\vec{E}=-\dfrac{\partial\vec{B}}{\partial t}\quad\Rightarrow\quad\vec{k}\wedge\vec{E}=\omega\mu\vec{H}$

$\Rightarrow\quad\vec{k}\wedge\left(\vec{k}\wedge\vec{E}\right)=\omega\mu\vec{k}\wedge\vec{H}=-\omega\mu\left(-i\sigma+\omega\varepsilon\right)\vec{E}$

$\Rightarrow\quad\left(\vec{k}\cdot\vec{E}\right)\vec{k}-k^2\vec{E}=-\omega\mu\left(-i\sigma+\omega\varepsilon\right)\vec{E}$

$\Rightarrow\quad k^2=\omega\mu\left(-i\sigma+\omega\varepsilon\right)$ since $\vec{k}\cdot\vec{E}=0$

For a good conductor D >> 1, $\sigma\gg\omega\varepsilon,\quad k^2\approx-i\omega\mu\sigma\quad\Rightarrow\quad k\approx\sqrt{\dfrac{\omega\mu\sigma}{2}}\left(1-i\right)$

Wave form is $\exp\left[i\left(\omega t-\dfrac{x}{\delta}\right)\right]\exp\left(-\dfrac{x}{\delta}\right),\quad k=\dfrac{1}{\delta}\left(1-i\right)$

copper.mov water.mov

where $\delta=\sqrt{\dfrac{2}{\omega\mu\sigma}}$ is the skin - depth

Charge Density in a Conducting Material

- Inside a conductor (Ohm's law) $\vec{j} = \sigma \vec{E}$
- Continuity equation is

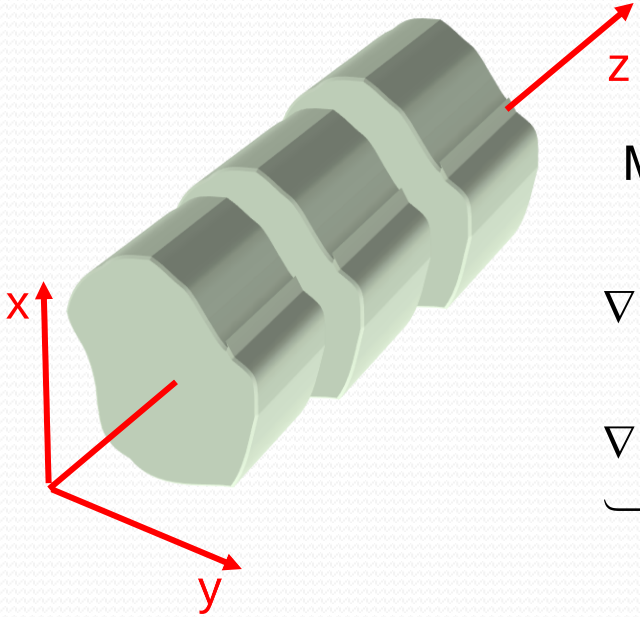
$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0$$

$$\Leftrightarrow \quad \frac{\partial \rho}{\partial t} + \sigma \nabla \cdot \vec{E} = 0 = \frac{\partial \rho}{\partial t} + \frac{\sigma}{\varepsilon} \rho$$

- Solution is $\rho = \rho_0 e^{-\sigma t/\varepsilon}$

So charge density decays exponentially with time. For a very good conductor, charges flow instantly to the surface to form a surface charge density and (for time varying fields) a surface current. Inside a perfect conductor ($\sigma \to \infty$) ***E*=*H*=0**

Maxwell's Equations in a Uniform Perfectly Conducting Guide



Hollow metallic cylinder with perfectly conducting boundary surfaces

Maxwell's equations with time dependence *exp(iωt)* are:

$$\nabla \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -i\omega\mu\vec{H} \qquad \Rightarrow \qquad \nabla^2\vec{E} = \nabla\left(\nabla\cdot\vec{E}\right) - \nabla\wedge\left(\nabla\wedge\vec{E}\right)$$

$$= i\omega\mu\nabla\wedge\vec{H}$$

$$\nabla \wedge \vec{H} = \frac{\partial \vec{D}}{\partial t} = i\omega\varepsilon\vec{E} \qquad = -\omega^2\varepsilon\mu\vec{E}$$

$$\left(\nabla^2 + \omega^2\mu\varepsilon\right)\begin{Bmatrix}\vec{E}\\ \vec{H}\end{Bmatrix} = 0$$

Assume $\vec{E}(x,y,z,t) = \vec{E}(x,y)e^{(i\omega t-\gamma z)}$

$\vec{H}(x,y,z,t) = \vec{H}(x,y)e^{(i\omega t-\gamma z)}$

Then $\left[\nabla_t^2 + (\omega^2\varepsilon\mu + \gamma^2)\right]\begin{Bmatrix}\vec{E}\\ \vec{H}\end{Bmatrix} = 0$

γ is the propagation constant

Can solve for the fields completely in terms of E_z and H_z

Special cases

- Transverse magnetic (TM modes):
  - $H_z=0$ everywhere, $E_z=0$ on cylindrical boundary
- Transverse electric (TE modes):
  - $E_z=0$ everywhere, $\frac{\partial H_z}{\partial n}=0$ on cylindrical boundary
- Transverse electromagnetic (TEM modes):
  - $E_z=H_z=0$ everywhere
  - requires

$$\gamma^2+\omega^2\varepsilon\mu=0 \quad \text{or} \quad \gamma=\pm i\omega\sqrt{\varepsilon\mu}$$

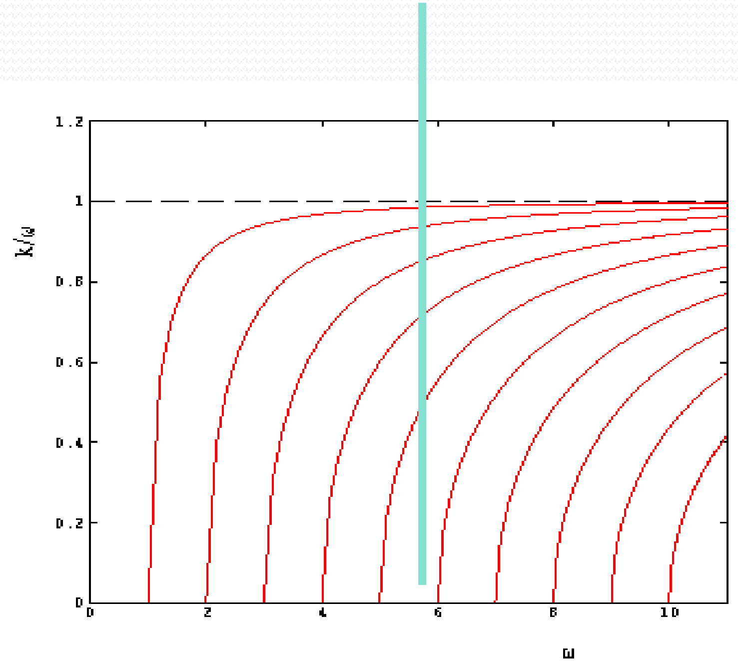
Cut-off frequency, ω_c

$$\gamma = \frac{n\pi}{a}\sqrt{1-\left(\frac{\omega}{\omega_c}\right)^2}, \quad E = A\sin\frac{n\pi x}{a}e^{i\omega t-\gamma z}, \quad \omega_c = \frac{n\pi}{a\sqrt{\varepsilon\mu}}$$

- $\omega<\omega_c$ gives real solution for γ, so attenuation only. No wave propagates: cut-off modes.
- $\omega>\omega_c$ gives purely imaginary solution for γ, and a wave propagates without attenuation.

$$\gamma = ik, \quad k = \sqrt{\varepsilon\mu}\left(\omega^2-\omega_c^2\right)^{1/2} = \omega\sqrt{\varepsilon\mu}\left(1-\frac{\omega_c^2}{\omega^2}\right)^{1/2}$$

- For a given frequency ω only a finite number of modes can propagate.



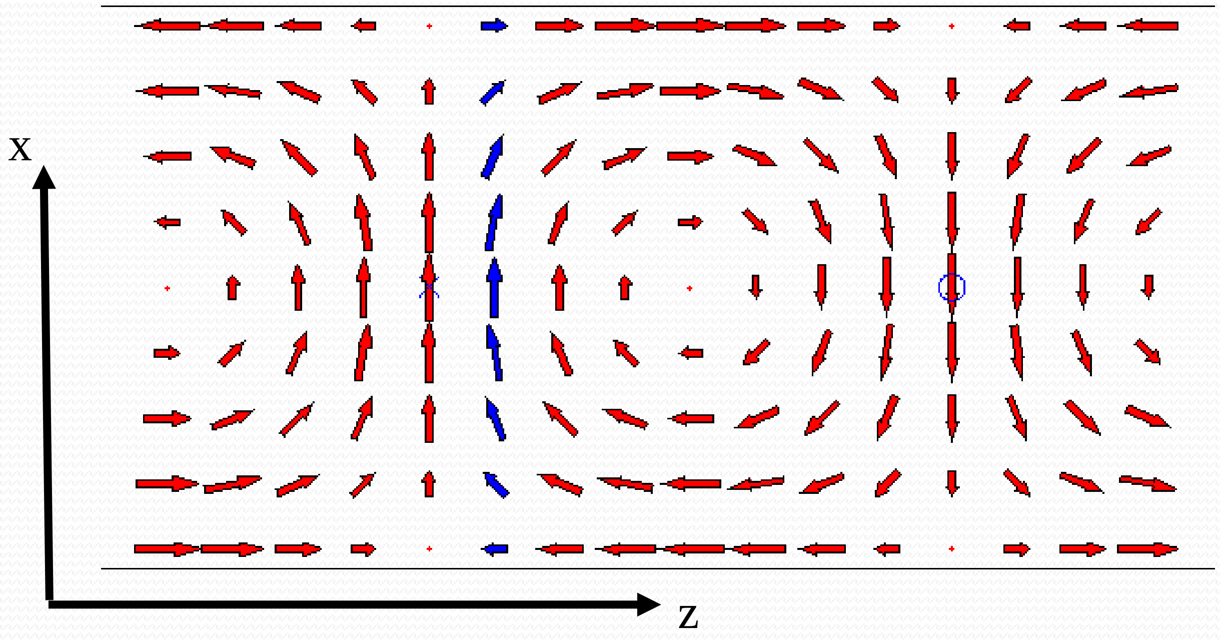
$$\omega > \omega_c = \frac{n\pi}{a\sqrt{\varepsilon\mu}} \quad \Rightarrow \quad n < \frac{a\omega}{\pi}\sqrt{\varepsilon\mu}$$

For given frequency, convenient to choose *a* s.t. only *n=1* mode occurs.

Propagated Electromagnetic Fields

From $\nabla \wedge \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$, assuming A is real,

$$\vec{H} = \frac{i}{\omega\mu}\nabla \wedge \vec{E} \quad \Rightarrow \quad \begin{cases} H_x = -\dfrac{Ak}{\omega\mu}\sin\left(\dfrac{n\pi x}{a}\right)\cos(\omega t - kz) \\ H_y = 0 \\ H_z = -\dfrac{A}{\omega\mu}\dfrac{n\pi}{a}\cos\left(\dfrac{n\pi x}{a}\right)\sin(\omega t - kz) \end{cases}$$



Phase and group velocities in the simple wave guide

Wave number: $k = \sqrt{\varepsilon\mu}\left(\omega^2 - \omega_c^2\right)^{1/2} < \omega\sqrt{\varepsilon\mu}$

Wavelength: $\lambda = \frac{2\pi}{k} > \frac{2\pi}{\omega\sqrt{\varepsilon\mu}}, \quad \text{the free} - \text{space wavelength}$

Phase velocity: $v_p = \frac{\omega}{k} > \frac{1}{\sqrt{\varepsilon\mu}},$

larger than free-space velocity

Group velocity:

$$k^2 = \varepsilon\mu\left(\omega^2 - \omega_c^2\right) \quad \Rightarrow \quad v_g = \frac{d\omega}{dk} = \frac{k}{\omega\varepsilon\mu} < \frac{1}{\sqrt{\varepsilon\mu}}$$

smaller than free-space velocity

Calculation of Wave Properties

- If a=3 cm, cut-off frequency of lowest order mode is

$$f_c = \frac{\omega_c}{2\pi} = \frac{1}{2a\sqrt{\varepsilon\mu}} \cong \frac{3\times10^8}{2\times0.03} \cong 5\,\mathrm{GHz}$$

$$\omega_c = \frac{n\pi}{a\sqrt{\varepsilon\mu}}$$

- At 7 GHz, only the n=1 mode propagates and

$$k = \sqrt{\varepsilon\mu}\left(\omega^2 - \omega_c^2\right)^{1/2} \cong 2\pi\left(7^2 - 5^2\right)^{1/2} \times 10^9 / 3\times10^8 \approx 103\,\mathrm{m}^{-1}$$

$$\lambda = \frac{2\pi}{k} \approx 6\,\mathrm{cm}$$

$$v_p = \frac{\omega}{k} \approx 4.3\times10^8\,\mathrm{ms}^{-1} > c$$

$$v_g = \frac{k}{\omega\varepsilon\mu} = 2.1\times10^8\,\mathrm{ms}^{-1} < c$$

Flow of EM energy along the simple guide

Fields ($\omega > \omega_c$) are:

$$E_x = E_z = 0, \quad E_y = A\sin\frac{n\pi x}{a}\cos(\omega t - kz)$$

$$H_x = -\frac{k}{\omega\mu}E_y, \quad H_y = 0, \quad H_z = -\frac{n\pi}{a\omega\mu}A\cos\frac{n\pi x}{a}\sin(\omega t - kz)$$

Time-averaged energy:

$$\text{Electric energy} \quad W_e = \frac{1}{4}\varepsilon\int_0^a \left|\vec{E}\right|^2 dx = \frac{1}{8}\varepsilon A^2 a$$

$$\text{Magnetic energy} \quad W_m = \frac{1}{4}\mu\int_0^a \left|\vec{H}\right|^2 dx = \frac{1}{8}\mu A^2 a\left\{\left(\frac{n\pi}{a\omega\mu}\right)^2 + \left(\frac{k}{\omega\mu}\right)^2\right\}$$

$$= W_e \quad \text{since} \quad k^2 + \frac{n^2\pi^2}{a^2} = \omega^2\varepsilon\mu$$

Total e/m energy density

$$W = \frac{1}{4}\varepsilon A^2 a$$

Group velocity:

$$v_g = \frac{d\omega}{d\beta} = \frac{1}{d\beta / d\omega}$$

$$\bar{E}(t,z) = E_0 \cos[(\omega + \Delta\omega)t - (\beta + \Delta\beta)z] + E_0 \cos[(\omega - \Delta\omega)t - (\beta - \Delta\beta)z]$$

$$= 2E_0 \cos(t\Delta\omega - z\Delta\beta)\cos(\omega t - \beta z)$$

Let $t\Delta\omega - z\Delta\beta$ =constant $\Rightarrow v_g = \frac{dz}{dt} = \frac{\Delta\omega}{\Delta\beta} = \frac{1}{\Delta\beta / \Delta\omega} = \frac{d\omega}{d\beta} = \frac{1}{d\beta / d\omega}$

Eg. Show that $v_g = v_p + \beta \frac{dv_p}{d\beta}$ **and** $v_g = v_p - \lambda \frac{dv_p}{d\lambda}$

(Proof) $v_p = \frac{\omega}{\beta}$, $\omega = v_p \beta$, $v_g = \frac{d\omega}{d\beta} = v_p + \beta \frac{dv_p}{d\beta}$

$\because\ \beta = \frac{2\pi}{\lambda}$, $\beta\lambda = 2\pi$, $\lambda d\beta + \beta d\lambda = 0 \Rightarrow \frac{\beta}{d\beta} = -\frac{\lambda}{d\lambda}$, $v_g = v_p - \lambda \frac{dv_p}{d\lambda}$

Poynting Vector

Poynting vector is $\vec{S} = \vec{E} \wedge \vec{H} = \left(E_y H_z, 0, -E_y H_x\right)$

Time-averaged: $\left\langle \vec{S} \right\rangle = \frac{1}{2}(0,0,1)\frac{kA^2}{\omega\mu}\sin^2\frac{n\pi x}{a}$

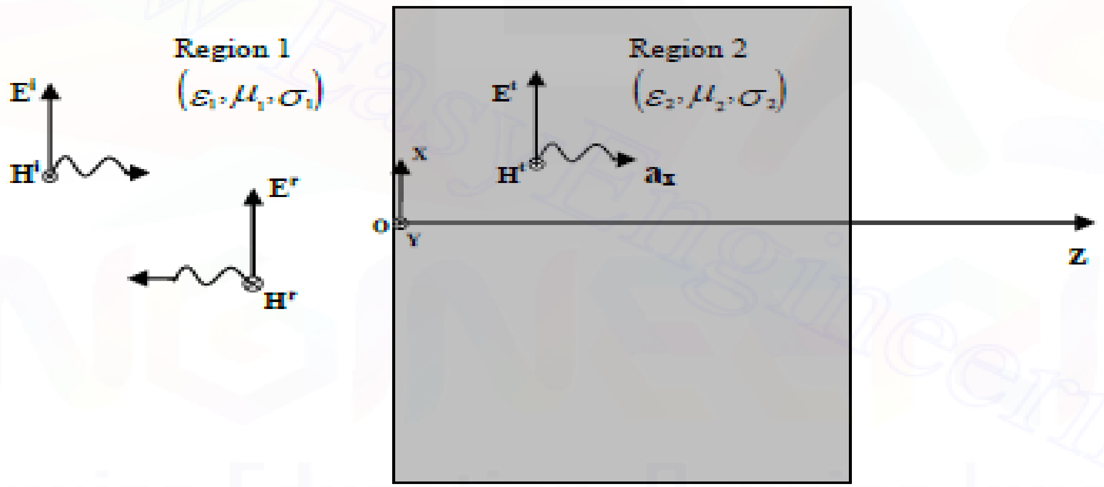
Integrate over *x*: $\left\langle S_z \right\rangle = \frac{1}{4}\frac{akA^2}{\omega\mu}$

Total e/m energy density

$$W = \frac{1}{4}\varepsilon A^2 a$$

So energy is transported at a rate: $\frac{\left\langle S_z \right\rangle}{W_e + W_m} = \frac{k}{\omega\varepsilon\mu} = v_g$

Electromagnetic energy is transported down the waveguide with the group velocity



Normal Incidence Plane Wave Reflection and Transmissions at Plane Boundary Between Two Conductive Media

The electric and magnetic fields related to the incident wave are given by the following:

$$\hat{E}^i_x = \hat{E}^+_{m1} e^{-\hat{\gamma}_1 z}$$

$$\hat{H}^i_y = \frac{\hat{E}^+_{m1}}{\hat{\eta}_1} e^{-\hat{\gamma}_1 z}$$

* Note: (i) incident, (m_1) medium 1, (γ_1) propagation constant in region 1, (η1) wave impedance in region 1, (z) direction of propagating wave

$$\gamma = \alpha + j\beta$$

With $$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^{2}}-1\right)}$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^{2}}+1\right)}$$

The wave impedance as defined in chapter 2 as the ratio between the electric and magnetic fields is

$$\frac{\hat{E}_x}{\hat{H}_y} = \hat{\eta} = \frac{\mu}{\left(\varepsilon - j\frac{\sigma}{\omega}\right)} = \frac{\sqrt{\frac{\mu}{\varepsilon}}}{\left[1+\left(\frac{\sigma}{\omega\varepsilon}\right)^{2}\right]^{1/4}} e^{j\frac{1}{2}\tan^{-1}\left(\frac{\sigma}{\omega\varepsilon}\right)}$$