UNIT – III RANDOM PROCESSES

PART - A

Problem 1. Define I & II order stationary Process
Solution:
I Order Stationary Process:
A random process is said to be stationary to order one if is first order density function does not change with a shift in time origin.

i.e., $f_X(x_1 : t_1) = f_X(x_1, t_1 + \delta)$ for any time t_1 and any real number δ.

i.e., $E[X(t)] = \overline{X} =$ Constant.

II Order Stationary Process:
A random process is said to be stationary to order two if its second-order density functions does not change with a shift in time origin.

i.e., $f_X(x_1, x_2 : t_1, t_2) = f_X(x_1, x_2 : t_1 + \delta, t_2 + \delta)$ for all t_1, t_2 and δ.

Problem 2. Define wide-sense stationary process
Solution:
A random process $X(t)$ is said to be wide sense stationary (WSS) process if the following conditions are satisfied

(i). $E[X(t)] = \mu$ i.e., mean is a constant

(ii). $R(\tau) = E[X(t_1)X(t_2)]$ i.e., autocorrelation function depends only on the time difference.

Problem 3. Define a strict sense stationary process with an example
Solution:
A random process is called a strongly stationary process (SSS) or strict sense stationary if all its statistical properties are invariant to a shift of time origin.
This means that $X(t)$ and $X(t+\delta)$ have the same statistics for any δ and any t
Example: Bernoulli process is a SSS process

Problem 4. Define n^{th} order stationary process, when will it become a SSS process?
Solution:
A random process $X(t)$ is said to be stationary to order n or n^{th} order stationary if its n^{th} order density function is invariant to a shift of time origin.

i.e., $f_X(x_1, x_2, ..., x_n, t_1, t_2, ..., t_n) = f_X(x_1, x_2, ..., x_n, t_1 + \delta, t_2 + \delta, ..., t_n + \delta)$ for all $t_1, t_2, ..., t_n$ & h.

A n^{th} order stationary process becomes a SSS process when $n \to \infty$.

Problem 5. When are two random process said to be orthogonal?
Solution:

Two process $\{X(t)\}$ & $\{Y(t)\}$ are said to be orthogonal, if $E\{X(t_1)Y(t_2)\} = 0.$

Problem 6. When are the process $\{X(t)\}$ & $\{Y(t)\}$ said to be jointly stationary in the wide sense?
Solution;

Two random process $\{X(t)\}$ & $\{Y(t)\}$ are said to be jointly stationary in the wide sense, if each process is individually a WSS process and $R_{XY}(t_1, t_2)$ is a function of $(t_2 - t_1)$ only.

7. Write the postulates of a poisson process?
Solution:

If $\{X(t)\}$ represents the number of occurrences of a certain event in $(0,t)$ then the discrete random process $\{X(t)\}$ is called the poisson process, provided the following postualates are satisfied

(i) $P[1 \; occumence \; in \; (t, t+\Delta t)] = \lambda\Delta t + o(\Delta t)$

(ii) $P[no \; occurrence \; in \; (t, t+\Delta t)] = 1 - \lambda\Delta t + o(\Delta t)$

(iii) $P[2 \; or \; more \; occurrences \; in \; (t, t+\Delta t)] = \lambda\Delta t + o(\Delta t)$

(iv) $X(t)$ is independent of the number of occurrences of the event in any interval prior and after the interval $(0,t)$.

(v) The probability that the event occurs in a specified number of times $(t_0, t_0 + t)$ depends only on t, but not on t_0.

8. When is a poisson process said to be homogenous?
Solution:

The rate of occurrence of the event λ is a constant, then the process is called a homogenous poisson process.

9. If the customers arrive at a bank according to a poisson process with a mean rate of 2 per minute, find the probability that, during an 1- minute interval no customer arrives.
Solution:

Here $\lambda = 2$, $t = 1$

$$\therefore P\{X(t) = n\} = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \; n = 0,1,2,...$$

Probability during 1-min interval, no customer arrives = $P\{X(t) = 0\} = e^{-2}$.

10. Define ergodic process.
Solution:
A random process $\{X(t)\}$ is said to be ergodic, if its ensemble average are equal to appropriate time averages.

11. Define a Gaussian process.
Solution:
A real valued random process $\{X(t)\}$ is called a Gaussian process or normal process, if the random variables $X(t_1), X(t_2),..., X(t_n)$ are jointly normal for every $n = 1, 2,...$ and for any set of $t_1, t_2,...$

The n^{th} order density of a Gaussian process is given by

$$f(x_1, x_2,..., x_n; t_1, t_2,..., t_n) = \frac{1}{(2\pi)^{n/2}|\Lambda|^{1/2}} \exp\left[-\frac{1}{2|\Lambda|}\sum_{i=1}^{n}\sum_{j=1}^{n}|\Lambda|_{ij}(x_i - \mu_i)(x_j - \mu_j)\right]$$

Where $\mu_i = E\{X(t_i)\}$ and Λ is the n^{th} order square matrix (λ_{ij}), where $\lambda_{ij} = C\{X(t_i), X(t_j)\}$ and $|\Lambda|_{ij}$ = Cofactor of λ_{ij} in $|\Lambda|$.

12. Define a Markov process with an example.
Solution:
If for $t_1 < t_2 < t_3 < ... < t_n < t$,

$$P\{X(t) \le x / X(t_1) = x_1, X(t_2) = x_2,..., X(t_n) = X_n\} = P\{X(t) \le x / X(t_n) = x_n\}$$

then the process $\{X(t)\}$ is called a markov process.
Example: The Poisson process is a Markov Process.

13. Define a Markov chain and give an example.
Solution:
If for all n,

$$P\{X_n = a_n / X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2},...X_0 = a_0\} = P\{X_n = a_n / X_{n-1} = a_{n-1}\},$$

then the process $\{x_n\}$, $n = 0,1,...$ is called a Markov chain.
Example: Poisson Process is a continuous time Markov chain.

Problem 14. What is a stochastic matrix? When is it said to be regular?
Solution: A sequence matrix, in which the sum of all the elements of each row is 1, is called a stochastic matrix. A stochastic matrix P is said to be regular if all the entries of P^m (for some positive integer m) are positive.

Problem 15. If the transition probability matrix of a markov chain is $\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ find the steady-state distribution of the chain.

Solution:

Let $\pi = (\pi_1, \pi_2)$ be the limiting form of the state probability distribution on stationary state distribution of the markov chain.

By the property of π, $\pi P = \pi$

i.e., $(\pi_1, \pi_2)\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = (\pi_1, \pi_2)$

$\frac{1}{2}\pi_2 = \pi_1$ ---------- (1)

$\pi_1 + \frac{1}{2}\pi_2 = \pi_2$ ------ (2)

Equation (1) & (2) are one and the same.

Consider (1) or (2) with $\pi_1 + \pi_2 = 1$, since π is a probability distribution.

$\pi_1 + \pi_2 = 1$

Using (1) , $\frac{1}{2}\pi_2 + \pi_2 = 1$

$\frac{3\pi_2}{2} = 1$

$\pi_2 = \frac{2}{3}$

$\pi_1 = 1 - \pi_2 = 1 - \frac{2}{3} = \frac{1}{3}$

$\pi_2 = 1 - \pi_1 = 1 - \frac{1}{3} = \frac{2}{3}$

$\therefore \pi_1 = \frac{1}{3}$ & $\pi_2 = \frac{2}{3}$.

PART-B

Problem 16. a). Define a random (stochastic) process. Explain the classification of random process. Give an example to each class.

Solution:

RANDOM PROCESS

A random process is a collection (orensemble) of random variables $\{X(s,t)\}$ that are functions of a real variable, namely time t where $s \in S$ (sample space) and $t \in T$ (Parameter set or index set).

CLASSIFICATION OF RANDOM PROCESS

Depending on the continuous on discrete nature of the state space S and parameter set T, a random process can be classified into four types:

(i). It both T & S are discrete, the random process is called a discrete random sequence.

Example: If X_n represents the outcome of the n^{th} toss of a fair dice, then $\{X_n, n \geq 1\}$ is a discrete random sequence, since $T = \{1, 2, 3, \ldots\}$ and $S = \{1, 2, 3, 4, 5, 6\}$.

(ii). If T is discrete and S is continuous, the random process is called a continuous random sequence.

Example: If X_n represents the temperature at the end n^{th} hour of a day, then $\{X_n, 1 \le n \le 24\}$ is a continuous random sequence since temperature can take any value is an interval and hence continuous.

(iii). If T is continuous and S is discrete, the random process is called a discrete random process.

Example: If $X(t)$ represents the number of telephone calls received in the interval $(0,t)$ then $\{X(t)\}$ random process, since $S = \{0,1,2,3,...\}$.

(iv). If both T and S are continuous, the random process is called a continuous random process f

Example: If $X(t)$ represents the maximum temperature at a place in the interval $(0,t)$ $\{X(t)\}$ is a continuous random process.

b). Consider the random process $X(t) = \cos(t + \varphi)$, where φ is uniformly distributed in the interval $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. Check whether the process is stationary or not.

Solution:

Since ϕ is uniformly distributed in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$f(\phi) = \frac{1}{\pi}, \quad -\frac{\pi}{2} < \phi < \frac{\pi}{2}$$

$$E[X(t)] = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} X(t) f(\phi)\, d\phi$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(t+\phi) \cdot \frac{1}{\pi}\, d\phi$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(t+\phi)\, d\phi$$

$$= \frac{1}{\pi}\left[\sin(t+\phi)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{2}{\pi}\cos t \ne \text{Constant.}$$

Since $E[X(t)]$ is a function of t, the random process $\{X(t)\}$ is not a stationary process.

Problem 17. a). Show that the process $\{X(t)\}$ whose probability distribution under certain conditions is given by $P\{X(t)=n\} = \begin{cases} \dfrac{(at)^{n-1}}{(1+at)^{n+1}}, & n = 1,2,\ldots \\ \dfrac{at}{1+at}, & n = 0 \end{cases}$ is evolutionary.

Solution:

The probability distribution is given by

$X(t)=n$	:	0	1	2	3	. . . .
$P(X(t)=n)$	:	$\dfrac{at}{1+at}$	$\dfrac{1}{(1+at)^2}$	$\dfrac{at}{(1+at)^3}$	$\dfrac{(at)^2}{(1+at)^4}$	. . . .

$$E[X(t)] = \sum_{n=0}^{\infty} np_n$$

$$= \frac{1}{(1+at)^2} + \frac{2at}{(1+at)^3} + \frac{3(at)^2}{(1+at)^4} + \ldots$$

$$= \frac{1}{(1+at)^2}\left\{1 + 2\left(\frac{at}{1+at}\right) + 3\left(\frac{at}{1+at}\right)^2 + \ldots\right\}$$

$$= \frac{1}{(1+at)^2}\left[1 - \frac{at}{1+at}\right]^{-2}$$

$$E[X(t)] = 1 = \text{Constant}$$

$$E[X^2(t)] = \sum_{n=0}^{\infty} n^2 p_n$$

$$= \sum_{n=1}^{\infty} n^2 \frac{(at)^{n-1}}{(1+at)^{n+1}} = \sum_{n=1}^{\infty}\left[n(n+1) - n\right]\frac{(at)^{n-1}}{(1+at)^{n+1}}$$

$$= \frac{1}{(1+at)^2}\left[\sum_{n=1}^{\infty} n(n+1)\left(\frac{at}{1+at}\right)^{n-1} - \sum_{n=1}^{\infty} n\left(\frac{at}{1+at}\right)^{n-1}\right]$$

$$= \frac{1}{(1+at)^2}\left[2\left[1 - \frac{at}{1+at}\right]^{-3} - \left[1 - \frac{at}{1+at}\right]^{-2}\right]$$

$$E[X^2(t)] = 1 + 2at \neq \text{Constant}$$

$$Var\{X(t)\} = E[X^2(t)] - [E(X(t))]^2$$

$$Var\{X(t)\} = 2at$$

$\therefore$ The given process $\{X(t)\}$ is evolutionary b). Examine whether the Poisson process $\{X(t)\}$ given by the probability law $P\{X(t)=n\} = \dfrac{e^{-\lambda t}(\lambda t)^n}{n!}$, $n = 0,1,2,...$ is evolutionary.

Solution:

$$E[X(t)] = \sum_{n=0}^{\infty} n p_n$$

$$= \sum_{n=0}^{\infty} n \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

$$= \sum_{n=1}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{(n-1)!}$$

$$= (\lambda t) e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!}$$

$$= (\lambda t) e^{-\lambda t} \left[1 + \frac{\lambda t}{1!} + \frac{(\lambda t)^2}{2!} + ...\right]$$

$$= (\lambda t) e^{-\lambda t} e^{\lambda t}$$

$$E[X(t)] = \lambda t$$

$E[X(t)] \neq$ Constant.

Hence the Poisson process $\{X(t)\}$ is evolutionary.

Problem 18. a). Show that the random process $X(t) = A\cos(\omega t + \theta)$ is WSS if A & ω are constants and θ is uniformly distributed random variable in $(0, 2\pi)$.

Solution:

Since θ is uniformly distributed random variable in $(0, 2\pi)$

$$f(\theta) = \begin{cases} \frac{1}{2\pi}, & 0 < 0 < 2\pi \\ 0 & , elsewhere \end{cases}$$

$$E[X(t)] = \int_0^{2\pi} X(t) f(\theta)\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2\pi} A cos(\omega t + \theta)\, d\theta$$

$$= \frac{A}{2\pi} \int_0^{2\pi} cos(\omega t + \theta)\, d\theta$$

$$= \frac{A}{2\pi} \left[sin(\omega t + \theta) \right]_0^{2\pi}$$

$$= \frac{A}{2\pi}\left[\sin(\omega t + 2\pi) - \sin(\omega t)\right]$$

$$= \frac{A}{2\pi}\left[\sin(\omega t) - \sin(\omega t)\right] \qquad \left[\because \sin(2\pi + \theta) = \sin\theta\right]$$

$$E\left[X(t)\right] = 0$$

$$R_{XX}(t_1, t_2) = E\left[X(t_1)X(t_2)\right]$$

$$= E\left[A^2\cos(\omega t_1 + \theta)\cos(\omega t_2 + \theta)\right]$$

$$= A^2 E\left[\frac{\cos(\omega(t_1 + t_2) + 2\theta) + \cos(\omega(t_1 - t_2))}{2}\right]$$

$$= \frac{A^2}{2}\int_0^{2\pi}\frac{1}{2\pi}\left[\cos(\omega(t_1 + t_2) + 2\theta) + \cos(\omega(t_1 - t_2))\right]d\theta$$

$$= \frac{A^2}{4\pi}\left[\frac{\sin\left[\omega(t_1 + t_2) + 2\theta\right]}{2} + \theta\cos\omega t\right]_0^{2\pi}$$

$$= \frac{A^2}{4\pi}\left[2\pi\cos\omega t\right]$$

$$= \frac{A^2}{2}\cos\omega t = \text{a function of time difference}$$

Since $E\left[X(t)\right] = \text{constant}$

$R_{XX}(t_1, t_2) =$ a function of time difference

$\therefore \{X(t)\}$ is a WSS.

b). Given a random variable y with characteristic function $\varphi(\omega) = E\left(e^{i\omega y}\right)$ and a random process define by $X(t) = \cos(\lambda t + y)$, show that $\{X(t)\}$ is stationary in the wide sense if $\varphi(1) = \varphi(2) = 0$.

Solution:

Given $\emptyset(1) = 0$

$$\Rightarrow E\left[\cos y + i\sin y\right] = 0$$

$$\therefore E\left[\cos y\right] = E\left[\sin y\right] = 0$$

Also $\emptyset(2) = 0$

$$\Rightarrow E\left[\cos 2y + i\sin 2y\right] = 0$$

$$\therefore E\left[\cos 2y\right] = E\left[\sin 2y\right] = 0$$

$$E\{X(t)\} = E\left[\cos(\lambda t + y)\right]$$

$$= E\left[\cos\lambda t\cos y - \sin\lambda t\sin y\right]$$

$$= \cos\lambda t\, E[\cos\lambda t] - \sin\lambda t\, E[\sin y] = 0$$

$$R_{XX}(t_1,t_2) = E[X(t_1)X(t_2)]$$

$$= E[\cos(\lambda t_1 + y)\cos(\lambda t_2 + y)]$$

$$= E\left[\frac{\cos(\lambda(t_1+t_2)+2y) + \cos(\lambda(t_1-t_2))}{2}\right]$$

$$= \frac{1}{2}E[\cos(\lambda(t_1+t_2)+2y) + \cos(\lambda(t_1-t_2))]$$

$$= \frac{1}{2}E[\cos\lambda(t_1+t_2)\cos 2y - \sin\lambda(t_1+t_2)\sin 2y + \cos(\lambda(t_1-t_2))]$$

$$= \frac{1}{2}\cos\lambda(t_1+t_2)E(\cos 2y) - \frac{1}{2}\sin\lambda(t_1+t_2)E(\sin 2y) + \frac{1}{2}\cos(\lambda(t_1-t_2))$$

$$= \frac{1}{2}\cos(\lambda(t_1-t_2))$$ = a function of time difference.

Since $E[X(t)]$ = constant

$R_{XX}(t_1,t_2)$ = a function of time difference

$\therefore \{X(t)\}$ is stationary in the wide sense .

Problem 19. a). If a random process $\{X(t)\}$ is defined by $\{X(t)\} = \sin(\omega t + Y)$ where Y is uniformly distributed in $(0, 2\pi)$. Show that $\{X(t)\}$ is WSS.

Solution:

Since y is uniformly distributed in $(0, 2\pi)$,

$$f(y) = \frac{1}{2\pi},\ 0 < y < 2\pi$$

$$E[X(t)] = \int_0^{2\pi} X(t) f(y)\, dy$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \sin(\omega t + y)\, dy$$

$$= \frac{1}{2\pi}\left[-\cos(\omega t + y)\right]_0^{2\pi}$$

$$= -\frac{1}{2\pi}[\cos(\omega t + 2\pi) - \cos\omega t] = 0$$

$$R_{XX}(t_1,t_2) = E[\sin(\omega t_1 + y)\sin(\omega t_2 + y)]$$

$$= E\left[\frac{\cos(\omega(t_1-t_2)) - \cos(\omega(t_1+t_2)+2y)}{2}\right]$$

$$= \frac{1}{2}E[\cos(\omega(t_1-t_2))] - \frac{1}{2}E[\cos(\omega(t_1+t_2)+2y)]$$

2 1 2 2 1 2

$$= \frac{1}{2}\cos\left(\omega\left(t_1 - t_2\right)\right) - \frac{1}{2}\int_0^{2\pi}\cos\left(\omega\left(t_1 + t_2\right) + 2y\right)\frac{1}{2\pi}dy$$

$$= \frac{1}{2}\cos\left(\omega\left(t_1 - t_2\right)\right) - \frac{1}{4\pi}\left[\frac{\sin\left(\omega\left(t_1 + t_2\right) + 2y\right)}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2}\cos\left(\omega\left(t_1 - t_2\right)\right) - \frac{1}{8\pi}\left[\sin\left(\omega\left(t_1 + t_2\right) + 2\pi\right) - \sin\omega\left(t_1 + t_2\right)\right]$$

$$= \frac{1}{2}\cos\left(\omega\left(t_1 - t_2\right)\right) \text{ is a function of time difence.}$$

$\therefore \{X(t)\}$ is WSS.

b). Verify whether the sine wave random process $X(t) = Y\sin\omega t$, Y is uniformly distributed in the interval $(-1,1)$ is WSS or not

Solution:

Since y is uniformly distributed in $(-1,1)$,

$$f(y) = \frac{1}{2}, \quad -1 < y < 1$$

$$E\left[X(t)\right] = \int_{-1}^{1} X(t)\, f(y)\,dy$$

$$= \int_{-1}^{1} y\sin\omega t\,\frac{1}{2}\,dy$$

$$= \frac{\sin\omega t}{2}\int_{-1}^{1} y\,dy$$

$$= \frac{\sin\omega t}{2}(0) = 0$$

$$R_{XX}\left(t_1, t_2\right) = E\left[X\left(t_1\right)X\left(t_2\right)\right]$$

$$= E\left[y^2\sin\omega t_1\sin\omega t_2\right]$$

$$= E\left[y^2\,\frac{\cos\omega\left(t_1 - t_2\right) - \cos w\left(t_1 + t_2\right)}{2}\right]$$

$$= \frac{\cos\omega\left(t_1 - t_2\right) - \cos w\left(t_1 + t_2\right)}{2}E\left(y^2\right)$$

$$= \frac{\cos\omega\left(t_1 - t_2\right) - \cos w\left(t_1 + t_2\right)}{2}\int_{-1}^{1} y^2 f(y)\,dy$$

$$= \frac{\cos\omega\left(t_1 - t_2\right) - \cos w\left(t_1 + t_2\right)}{2}\,\frac{1}{2}\int_{-1}^{1} y^2\,dy$$

$$= \frac{\cos\omega(t_1 - t_2) - \cos w(t_1 + t_2)}{4}\left(\frac{y^3}{3}\right)_{-1}^{1}$$

$$= \frac{\cos\omega(t_1 - t_2) - \cos w(t_1 + t_2)}{4}\left(\frac{2}{3}\right)$$

$$= \frac{\cos\omega(t_1 - t_2) - \cos w(t_1 + t_2)}{6}$$

$R_{XX}(t_1, t_2) \neq$ a function of time difference alone.

Hence it is not a WSS Process.

Problem 20. a). Show that the process $X(t) = A\cos\lambda t + B\sin\lambda t$ (where A & B are random variables) is WSS, if (i) $E(A) = E(B) = 0$ (ii) $E(A^2) = E(B^2)$ and (iii) $E(AB) = 0$.

Solution:

Given $X(t) = A\cos\lambda t + B\sin\lambda t$, $E(A) = E(B) = 0$, $E(AB) = 0$,

$E(A^2) = E(B^2) = k(say)$

$E[X(t)] = \cos\lambda t\, E(A) + \sin\lambda t\, E(B)$

$E[X(t)] = 0 =$ is a constant. $\because E(A) = E(B) = 0$

$R(t_1, t_2) = E\{X(t_1)X(t_2)\}$

$= E\{(A\cos\lambda t_1 + B\sin\lambda t_1)(A\cos\lambda t_2 + B\sin\lambda t_2)\}$

$= E(A^2)\cos\lambda t_1\cos\lambda t_2 + E(B^2)\sin\lambda t_1\sin\lambda t_2 + E(AB)[\sin\lambda t_1\cos\lambda t_2 + \cos\lambda t_1\sin\lambda t_2]$

$= E(A^2)\cos\lambda t_1\cos\lambda t_2 + E(B^2)\sin\lambda t_1\sin\lambda t_2 + E(AB)\sin\lambda(t_1 + t_2)$

$= k(\cos\lambda t_1\cos\lambda t_2 + \sin\lambda t_1\sin\lambda t_2)$

$= k\cos\lambda(t_1 - t_2) =$ is a function of time difference.

$\therefore \{X(t)\}$ is WSS.

b). If $X(t) = Y\cos t + Z\sin t$ for all t & where Y & Z are independent binary random variables. Each of which assumes the values – 1 & 2 with probabilities $\frac{2}{3}$ & $\frac{1}{3}$ respectively, prove that $\{X(t)\}$ is WSS.

Solution:

Given

$Y = y$	:	-1	2
$P(Y = y)$	:	$\frac{2}{3}$	$\frac{1}{3}$

$E(Y) = E(Z) = -1 \times \frac{2}{3} + 2 \times \frac{1}{3} = 0$

$E(Y^2) = E(Z^2) = (-1)^2 \times \frac{2}{3} + (2)^2 \times \frac{1}{3}$

$E(Y^2) = E(Z^2) = \frac{2}{3} + \frac{4}{3} = \frac{6}{3} = 2$

Since Y & Z are independent

$E(YZ) = E(Y)E(Z) = 0$ ---- (1)

Hence $E[X(t)] = E[ycost + zsint]$

$= E[y]cost + E[z]sint$

$E[X(t)] = 0 =$ is a constant. $\quad [\because E(y) = E(z) = 0]$

$R_{XX}(t_1,t_2) = E[X(t_1)X(t_2)]$

$= E[(ycost_1 + zsint_1)(ycost_2 + zsint_2)]$

$= E[y^2cost_1cost_2 + yzcost_1sint_2 + zysint_1cost_2 + z^2sint_1sint_2]$

$= E(y^2)cost_1cost_2 + E[yz]cost_1sint_2 + E[zy]sint_1cost_2 + E[z^2]sint_1sint_2$

$= E(y^2)cost_1cost_2 + E(z^2)sint_1sint_2$

$= 2[cost_1cost_2 + sint_1sint_2] \quad [\because E(y^2) = E(z^2) = 2]$

$= 2cos(t_1 - t_2) =$ is a function of time difference.

$\therefore \{X(t)\}$ is WSS.

Problem 21. a). Check whether the two random process given by $X(t) = A\cos\omega t + B\sin\omega t$ & $Y(t) = B\cos\omega t - A\sin\omega t$. Show that $X(t)$ & $Y(t)$ are jointly WSS if A & B are uncorrelated random variables with zero mean and equal variance random variables are jointly WSS.

Solution:

Given $E(A) = E(B) = 0$

$Var(A) = Var(B) = \sigma^2$

$\therefore E(A^2) = E(B^2) = \sigma^2$

As $A \& B$ uncorrelated are $E(AB) = E(A)E(B) = 0$.

$E[X(t)] = E[Acos\omega t + Bsin\omega t]$

$= E(A)cos\omega t + E(B)sin\omega t = 0$

$E[X(t)] = 0 =$ is a constant.

$R_{XX}(t_1,t_2) = E[X(t_1)X(t_2)]$

$$= E\left[\left(A\cos wt_1 + B\sin wt_2\right)\left(A\cos wt_2 + B\sin wt_2\right)\right]$$

$= E\left[A^2 \cos wt_1 \cos wt_2 + AB\cos wt_1 \sin wt_2 + BA\sin wt_1 \cos wt_2 + B^2 \sin wt_1 \sin wt_2\right]$

$= \cos wt_1 \cos wt_2 \, E\left[A^2\right] + \cos wt_1 \sin wt_2 \, E[AB] + \sin wt_1 \cos wt_2 \, E[BA] + \sin wt_1 \sin wt_2 \, E\left[B^2\right]$

$= \sigma^2 \left[\cos wt_1 \cos wt_2 + \sin wt_1 \sin wt_2\right]$

$= \sigma^2 \cos w\left(t_1 - t_2\right) \qquad \left[\because E\left(A^2\right) = E\left(B^2\right) = \sigma^2 \ \& \ E(AB) = E(BA) = 0\right]$

$R_{XX}\left(t_1, t_2\right)$ =is a function of time difference.

$E\left[Y(t)\right] = E\left[B\cos\omega t - A\sin\omega t\right]$

$= E(B)\cos\omega t - E(A)\sin\omega t = 0$

$R_{YY}\left(t_1, t_2\right) = E\left[\left(B\cos\omega t_1 - A\sin\omega t_1\right)\left(B\cos\omega t_2 - A\sin\omega t_2\right)\right]$

$= E\left[B^2\cos\omega t_1 \cos\omega t_2 - BA\cos\omega t_1 \sin\omega t_2 - AB\sin\omega t_1 \cos\omega t_2 + A^2 \sin\omega t_1 \sin\omega t_2\right]$

$= E\left(B^2\right)\cos\omega t_1 \cos\omega t_2 - E(BA)\cos\omega t_1 \sin\omega t_2 - E\left(A^2B\right)\sin\omega t_1 \cos\omega t_2 + E\left(A^2\right)\sin\omega t_1 \sin\omega t_2$

$= \sigma^2 \cos\omega\left(t_1 - t_2\right) \qquad \left[\because E\left(A^2\right) = E\left(B^2\right) = \sigma^2 \ \& \ E(AB) = E(BA) = 0\right]$

$R_{YY}\left(t_1, t_2\right)$ =is a function of time difference.

$R_{XY}\left(t_1, t_2\right) = E\left[X\left(t_1\right)Y\left(t_2\right)\right]$

$= E\left[\left(A\cos\omega t_1 + B\sin\omega t_1\right)\left(B\cos\omega t_2 + A\sin\omega t_2\right)\right]$

$= E\left[AB\cos\omega t_1 \cos\omega t_2 - A^2\cos\omega t_1 \sin\omega t_2 + B^2 \sin\omega t_1 \cos\omega t_2 - BA\sin\omega t_1 \sin\omega t_2\right]$

$= \sigma^2\left[\sin\omega t_1 \cos\omega t_2 - \cos\omega t_1 \sin\omega t_2\right]$

$= \sigma^2 \sin\omega\left(t_1 - t_2\right) \qquad \left[\because E\left(A^2\right) = E\left(B^2\right) = \sigma^2 \ \& \ E(AB) = E(BA) = 0\right]$

$R_{XY}\left(t_1, t_2\right)$ = is a function of time difference.

Since $\{X(t)\}$ & $\{Y(t)\}$ are individually WSS & also $R_{XY}\left(t_1, t_2\right)$ is a function of time difference.

$\therefore$ The two random process $\{X(t)\}$ & $\{Y(t)\}$ are jointly WSS.

b). Write a note on Binomial process.

Solution:

Binomial Process can be defined as a sequence of partial sums $\{S_n / n = 1, 2, ...\}$ Where $S_n = X_1 + X_2 + ... + X_n$ Where X_i denotes 1 if the trial is success or 0 if the trial is failure.

As an example for a sample function of the binomial random process with $\left(x_1, x_2, ...\right) = (1,1,0,0,1,0,1,...)$ is $\left(s_1, s_2, s_3, ...\right) = (1,2,2,2,3,3,4,...)$. The process increments by 1 only at the discrete times $t_i \; t_i = iT$, $i = 1, 2, ...$

Properties

(i). Binomial process is Markovian

(ii). S_n is a binomial random variable so, $P(S_n = m) = nC_m p^m q^{n-m}$, $E[S_n] = np$ & $var[S_n] = np(1-p)$

(iii) The distribution of the number of slots m_i between i^{th} and $(i-1)^{th}$ arrival is geometric with parameter p starts from 0. The random variables m_i, $i = 1, 2, ...$ are mutually independent.

The geometric distribution is given by $p(1-p)^{i-1}, i = 1, 2, ...$

(iv) The binomial distribution of the process approaches poisson when n is large and p is small.

Problem 22. a). Describe Poisson process & show that the Poisson process is Markovian.

Solution:

If $\{X(t)\}$ represents the number of occurrences of a certain event in $(0,t)$ then the discrete random process $\{X(t)\}$ is called the Poisson process, provided the following postulates are satisfied

(i) $P[1 \text{ occurrence in } (t, t+\Delta t)] = \lambda\Delta t + o(\Delta t)$

(ii) $P[\text{no occurrence in } (t, t+\Delta t)] = 1 - \lambda\Delta t + o(\Delta t)$

(iii) $P[2 \text{ or more occurrences in } (t, t+\Delta t)] = o(\Delta t)$

(iv) $X(t)$ is independent of the number of occurrences of the event in any interval prior and after the interval $(0,t)$.

(v) The probability that the event occurs in a specified number of times $(t_0, t_0 + t)$ depends only on t, but not on t_0.

Consider

$$P\left[X(t_3) = n_3 / X(t_2) = n_2, X(t_1) = n_1\right] = \frac{P\left[X(t_1) = n_1, X(t_2) = n_2, X(t_3) = n_3\right]}{P\left[X(t_1) = n_1, X(t_2) = n_2\right]}$$

$$= \frac{e^{-\lambda t_3}\lambda^{n_3} t_1^{n_1}(t_2 - t_1)^{n_2-n_1}(t_3 - t_2)^{n_3-n_2}}{n_1!(n_2 - n_1)!(n_3 - n_2)!}$$

$$= \frac{e^{-\lambda t_2}\lambda^{n_2} t_1^{n_1}(t_2 - t_1)^{n_2-n_1}}{n_1!(n_2 - n_1)!}$$

$$= \frac{e^{-\lambda(t_3-t_2)}\lambda^{n_3-n_2}(t_3 - t_2)^{n_3-n_2}}{(n_3 - n_2)!}$$

$$P\left[X(t_3) = n_3 / X(t_2) = n_2, X(t_1) = n_1\right] = P\left[X(t_3) = n_3 / X(t_2) = n_2\right]$$

This means that the conditional probability distribution of $X(t_3)$ given all the past values $X(t_1) = n_1$, $X(t_2) = n_2$ depends only on the most recent values $X(t_2) = n_2$.

i.e., The Poisson process possesses Markov property.

b). State and establish the properties of Poisson process.
Solution:
(i). Sum of two independent poisson process is a poisson process.
Proof:
The moment generating function of the Poisson process is

$$M_{X(t)}(u) = E\left[e^{uX(t)}\right]$$

$$= \sum_{x=0}^{\infty} e^{ux} P\left[X(t) = x\right]$$

$$= \sum_{x=0}^{\infty} e^{ux} \frac{e^{-\lambda t}(\lambda t)^x}{x!}$$

$$= e^{-\lambda t}\left[1 + \frac{e^u(\lambda t)^1}{1!} + \frac{e^{2u}(\lambda t)^2}{2!} + \dots\right]$$

$$= e^{-\lambda t} e^{\lambda t e^u}$$

$$M_{X(t)}(u) = e^{\lambda t(e^u - 1)}$$

Let $X_1(t)$ and $X_2(t)$ be two independent Poisson processes

$\therefore$ Their moment generating functions are,

$$M_{X_1(t)}(u) = e^{\lambda_1 t(e^u-1)} \text{ and } M_{X_2(t)}(u) = e^{\lambda_2 t(e^u-1)}$$

$$\therefore M_{X_1(t)+X_2(t)}(u) = M_{X_1(t)}(u) M_{X_2(t)}(u)$$

$$= e^{\lambda_1 t(e^u-1)} e^{\lambda_2 t(e^u-1)}$$

$$= e^{(\lambda_1+\lambda_2)t(e^u-1)}$$

$\therefore$ By uniqueness of moment generating function, the process $\{X_1(t) + X_2(t)\}$ is a Poisson process with occurrence rate $(\lambda_1 + \lambda_2)$ per unit time.

(ii). Difference of two independent poisson process is not a poisson process.
Proof:
Let $X(t) = X_1(t) - X_2(t)$

$$E\{X(t)\} = E\{X_1(t)\} - E\{X_2(t)\}$$

$$= (\lambda_1 - \lambda_2)t$$

$$E\{X^2(t)\} = E\{X_1^2(t)\} + E\{X_2^2(t)\} - 2E\{X_1(t)\}E\{X_2(t)\}$$

$$= (\lambda_1^2 t^2 + \lambda_1 t) + (\lambda_2^2 t^2 + \lambda_2 t) - 2(\lambda_1 t)(\lambda_2 t)$$

$$= (\lambda_1 + \lambda_2)t + (\lambda_1 - \lambda_2)^2 t^2$$

$$\neq (\lambda_1 - \lambda_2)t + (\lambda_1 - \lambda_2)^2 t^2$$

Recall that $E\{X^2(t)\}$ for a poisson process $\{X(t)\}$ with parameter λ is given by

$E\left\{X^2(t)\right\} = \lambda t + \lambda^2 t^2$

$\therefore X_1(t) - X_2(t)$ is not a poisson process.

(iii). The inter arrival time of a poisson process i.e., with the interval between two successive occurrences of a poisson process with parameter λ has an exponential distribution with mean $\frac{1}{\lambda}$.

Proof:

Let two consecutive occurrences of the event be E_i & E_{i+1}.

Let E_i take place at time instant t_i and T be the interval between the occurrences of E_i E_{i+1}.

Thus T is a continuous random variable.

$$P(T > t) = P\{Interval\ between\ occurrence\ of\ E_i\ and\ E_{i+1}\ exceeds\ t\}$$
$$= P\{E_{i+1}\ does\ not\ occur\ upto\ the\ instant\ (t_i + t)\}$$
$$= P\{No\ event\ occurs\ in\ the\ interval\ (t_i, t_i + t)\}$$
$$= P\{X(t) = 0\} = P_0(t)$$
$$= e^{-\lambda t}$$

$\therefore$The cumulative distribution function of T is given by

$$F(t) = P\{T \le t\} = 1 - e^{-\lambda t}$$

$\therefore$The probability density function is given by

$$f(t) = \lambda e^{-\lambda t},\ (t \ge 0)$$

Which is an exponential distribution with mean $\frac{1}{\lambda}$.

Problem 23. a). If the process $\{N(t) : t \ge 0\}$ is a Poisson process with parameter λ, obtain $P[N(t) = n]$ and $E[N(t)]$.

Solution:

Let λ be the number of occurrences of the event in unit time.

Let $P_n(t)$ represent the probability of n occurrences of the event in the interval $(0, t)$.

i.e., $P_n(t) = P\{X(t) = n\}$

$$\therefore P_n(t + \Delta t) = P\{X(t + \Delta t) = n\}$$
$$= P\{n\ occurences\ in\ the\ time\ (0, t + \Delta t)\}$$
$$= P\begin{cases} n\ occurences\ in\ the\ interval\ (0,t)\ and\ no\ occurences\ in\ (t, t+\Delta t)\ or \\ n-1\ occurences\ in\ the\ interval\ (0,t)\ and\ 1\ occurences\ in\ (t, t+\Delta t)\ or \\ n-2\ occurences\ in\ the\ interval\ (0,t)\ and\ 2\ occurences\ in\ (t, t+\Delta t)\ or ... \end{cases}$$
$$= P_n(t)(1 - \lambda \Delta t) + P_{n-1}(t)\lambda \Delta t + 0 + ...$$

$$\therefore \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = \lambda\left\{P_{n-1}(t) - P_n(t)\right\}$$

Taking the limits as $\Delta t \to 0$

$$\frac{d}{dt} P_n(t) = \lambda\left\{P_{n-1}(t) - P_n(t)\right\} \text{ ------- } (1)$$

This is a linear differential equation.

$$\therefore P_n(t)\, e^{\lambda t} = \int_0^t \lambda P_{n-1}(t)\, e^{\lambda t}\, dt \text{ ---------------- } (2)$$

Now taking $n = 1$ we get

$$e^{\lambda t} P_1(t) = \lambda \int_0^t P_0(t)\, e^{\lambda t}\, dt \text{ ---------------- } (3)$$

Now, we have,

$$P_0(t+\Delta t) = P\{0 \text{ occurences in } (0, t+\Delta t)\}$$

$$= P\{0 \text{ occurences in } (0,t) \text{ and } 0 \text{ occurences in } (t, t+\Delta t)\}$$

$$= P_0(t)[1 - \lambda \Delta t]$$

$$P_0(t+\Delta t) - P_0(t) = -P_0(t)(\lambda \Delta t)$$

$$\frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} = -\lambda P_0(t)$$

$\therefore$ Taking limit $\Delta t \to 0$

$$\underset{\Delta t \to 0}{Lt} \frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} = -\lambda P_0(t)$$

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t)$$

$$\frac{dP_0(t)}{P_0(t)} = -\lambda\, dt$$

$$\log P_0(t) = -\lambda t + c$$

$$P_0(t) = e^{-\lambda t + c}$$

$$P_0(t) = e^{-\lambda t} e^{c}$$

$$P_0(t) = e^{-\lambda t} A \text{ ---------------------- } (4)$$

Putting $t = 0$ we get

$$P_0(0) = e^0 A = A$$

i.e., $A = 1$

$\therefore$ (4) we have

$$P_0(t) = e^{-\lambda t}$$

$\therefore$ substituting in (3) we get

$$e^{\lambda t} P_1(t) = \lambda \int_0^t e^{-\lambda t} e^{\lambda t}\, dt$$

$$= \lambda \int_0^t dt = \lambda t$$

$$P_1(t) = e^{-\lambda t} \lambda t$$

Similarly $n = 2$ in (2) we have,

$$P_2(t) e^{\lambda t} = \lambda \int_0^t P_1(t) e^{\lambda t} dt$$

$$= \lambda \int_0^t e^{-\lambda t} \lambda t e^{\lambda t} dt$$

$$= \lambda^2 \left[\frac{t^2}{2} \right]$$

$$P_2(t) e^{\lambda t} = \frac{e^{-\lambda t} (\lambda t)^2}{2!}$$

Proceeding similarly we have in general

$$P_n(t) = P\{X(t) = n\} = \frac{e^{-\lambda t} (\lambda t)^n}{n!}, \; n = 0,1,...$$

Thus the probability distribution of $X(t)$ is the Poisson distribution with parameter λt.

$$E\left[X(t)\right] = \lambda t.$$

.b). Find the mean and autocorrelation and auto covariance of the Poisson process.

Solution:

The probability law of the poisson process $\{X(t)\}$ is the same as that of a poisson distribution with parameter λt

$$E\left[X(t)\right] = \sum_{n=0}^{\infty} nP\left(X(t) = n\right)$$

$$= \sum_{n=0}^{\infty} n \frac{e^{-\lambda t} (\lambda t)^n}{n!}$$

$$= e^{-\lambda t} \sum_{n=1}^{\infty} \frac{\lambda t (\lambda t)^{n-1}}{(n-1)!}$$

$$= \lambda t e^{-\lambda t} e^{\lambda t}$$

$$E\left[X(t)\right] = \lambda t$$

$$Var\left[X(t)\right] = E\left[X^2(t)\right] - E\left[X(t)\right]^2$$

$$E\left\{X^2(t)\right\} = \sum_{n=0}^{\infty} n^2 P\left(X(t) = n\right)$$

$$= \sum_{n=0}^{\infty} \left(n^2 - n + n\right) \frac{e^{-\lambda t} (\lambda t)^n}{n!}$$

$$= \sum_{n=0}^{\infty} n(n-1)\frac{e^{-\lambda t}(\lambda t)^n}{n!} + \sum_{n=0}^{\infty} n\frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

$$= \sum_{n=0}^{\infty} n(n-1)\frac{e^{-\lambda t}(\lambda t)^n}{n!} + \lambda t$$

$$= e^{-\lambda t}\left[\frac{(\lambda t)^2}{1} + \frac{(\lambda t)^3}{1!} + \frac{(\lambda t)^4}{2!} + \dots\right] + \lambda t$$

$$= (\lambda t)^2 e^{-\lambda t} e^{\lambda t} + \lambda t$$

$$= (\lambda t)^2 + \lambda t$$

$$E\{X^2(t)\} = \lambda t + \lambda^2 t^2$$

$$\therefore Var\left[X(t)\right] = \lambda t$$

$$R_{XX}(t_1,t_2) = E\{X(t_1)X(t_2)\}$$

$$= E\{X(t_1)\left[X(t_2) - X(t_1) + X(t_1)\right]\}$$

$$= E\{X(t_1)\left[X(t_2) - X(t_1)\right] + E\left[X^2(t_1)\right]\}$$

$$= E\left[X(t_1)\right] E\left[X(t_2) - X(t_1)\right] + E\left[X^2(t_1)\right]$$

Since $\{X(t)\}$ is a process of independent increments.

$$= \lambda t_1\left[\lambda(t_2 - t_1)\right] + \lambda t_1 + \lambda t_1^2 \quad if\ t_2 \ge t_1 \quad (by-(1))$$

$$= \lambda^2 t_1 t_2 + \lambda t_1 \quad if\ t_2 \ge t_1$$

$$R_{XX}(t_1,t_2) = \lambda^2 t_1 t_2 + \lambda \min(t_1,t_2)$$

Auto Covariance

$$C_{XX}(t_1,t_2) = R_{XX}(t_1,t_2) - E\{X(t_1)\}E\{X(t_2)\}$$

$$= \lambda^2 t_1 t_2 + \lambda t_1 - \lambda^2 t_1 t_2$$

$$= \lambda t_1, \quad if\ t_2 \ge t_1$$

$$= \lambda \min(t_1,t_2)$$

Problem 24. a). Prove that the random process $X(t) = A\cos(\omega t + \theta)$. Where A, ω are constants θ is uniformly distributed random variable in $(0, 2\pi)$ is ergodic.

Solution:

Since θ is uniformly distribution in $(0, 2\pi)$.

$$f(\theta) = \frac{1}{2\pi},\ 0 < \theta < 2\pi$$

Ensemble average same,

$$E\left[X(t)\right] = \int_0^{2\pi} \frac{1}{2\pi}\cos(\omega t + \theta)\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(\cos\omega t\cos\theta - \sin\omega t\sin\theta\right)d\theta$$

$$= \frac{1}{2\pi}\left[\cos\omega t\sin\theta + \sin\omega t\cos\theta\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\sin\omega t - \sin\omega t\right] = 0$$

$$E\left[X(t)\right] = 0$$

$$R_{XX}(t_1,t_2) = E\left[X(t_1)X(t_2)\right]$$

$$= E\left[\cos(\omega t_1+\theta)\cos(\omega t_2+\theta)\right]$$

$$= \frac{1}{2}E\left[\cos(\omega t_1+\omega t_2+2\theta)+\cos(\omega t_1-\omega t_2)\right]$$

$$= \frac{1}{2}\int_0^{2\pi}\frac{1}{2\pi}\left[\cos(\omega t_1+\omega t_2+2\theta)+\cos(\omega t_1-\omega t_2)\right]d\theta$$

$$= \frac{1}{4\pi}\left[\frac{\sin(\omega t_1+\omega t_2+2\theta)+\theta\cos(\omega(t_1-t_2))}{2}\right]_0^{2\pi}$$

$$= \frac{2\pi}{4\pi}\cos\left(\omega(t_1-t_2)\right)$$

$$R_{XX}(t_1,t_2) = \frac{1}{2}\cos\left(\omega(t_1-t_2)\right)$$

The time average can be determined by

$$\overline{X(t)} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\cos(\omega t+\theta)\,dt$$

$$= \lim_{T\to\infty}\frac{1}{2T}\left[\frac{\sin(\omega t+\theta)}{\omega}\right]_{-T}^{T}$$

$$= \lim_{T\to\infty}\frac{1}{2T\omega}\left[\sin(\omega T+\theta)-\sin(-\omega T+\theta)\right]$$

$$\overline{X(t)} = 0\ \left[As\ T->\infty\right]$$

The time auto correlation function of the process,

$$\underset{T\to\infty}{Lt}\frac{1}{2T}\int_{-T}^{T}X(t)X(t+\tau)\,dt$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\cos(\omega t+\theta)\cos(\omega t+\omega\tau+\theta)\,dt$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left[\frac{\cos(\omega t+\omega\tau+2\theta)+\cos\omega\tau}{2}\right]dt$$

$$= \lim_{T \to \infty} \frac{1}{4T} \left[\frac{\sin(2\omega t + \omega t + 2\theta)}{2} + t\cos\omega t \right]_{-T}^{T}$$

$$= \lim_{T \to \infty} \frac{1}{4T} \left[\frac{\sin(2\omega t + \omega t + 2\theta) + \sin(-2\omega T + \omega\tau + 2\theta)}{2\omega} + 2T\cos\omega\tau \right]$$

$$= \frac{\cos\omega\tau}{2}$$

Since the ensemble average=time average the given process is ergodic.

b). If the WSS process $\{X(t)\}$ is given by $X(t) = 10\cos(100t + \theta)$ where θ is uniformly distributed over $(-\pi, \pi)$ prove that $\{X(t)\}$ is correlation ergodic.

Solution:

To Prove $\{X(t)\}$ is correlation ergodic it is enough to show that when

$$\underset{T \to \infty}{Lt} \frac{1}{2T} \int_{-T}^{T} X(t)X(t+\tau)\,dt = R_{XX}(\tau)$$

$$R_{XX}(\tau) = E[X(t)X(t+\tau)]$$

$$= E[10\cos(100t + \theta)10\cos(100t + 100\tau + \theta)]$$

$$= E\left\{100\left[\frac{\cos t(200t + 100\tau + 2\theta) + \cos(100\tau)}{2}\right]\right\}$$

$$= 50\int_{-\pi}^{\pi} \frac{1}{2\pi}\left[\cos(200t + 100\tau + 2\theta) + \cos(100\tau)\right] d\theta$$

$$= \frac{50}{2\pi}\left[\frac{\sin(200t + 100\tau + 2\theta)}{2} + \theta\cos(100\tau)\right]_{-\pi}^{\pi}$$

$$R_{XX}(\tau) = 50\cos(100\tau)$$

$$\underset{T \to \infty}{Lt} \frac{1}{2T} \int_{-T}^{T} X(t)X(t+\tau)\,dt$$

$$= \underset{T \to \infty}{Lt} \frac{1}{2T} \int_{-T}^{T} \left[10\cos(100t + \theta)10\cos(100t + 100\tau + \theta)\right] dt$$

$$= \underset{T \to \infty}{Lt} \frac{25}{T} \int_{-T}^{T} \left[\cos(200t + 100\tau + 2\theta) + \cos(100\tau)\right] dt$$

$$= \underset{T \to \infty}{Lt} \frac{25}{T} \left[\frac{\sin(200t + 100\tau + 2\theta)}{200} + t\cos(100\tau)\right]_{-T}^{T}$$

$$\lim_{T \to \infty} \overline{X_T} = \lim_{T \to \infty} \frac{50}{T} T\cos(100\tau)$$

$= 50\cos(100\tau)$ = is a function of time difference

$\therefore \{X(t)\}$ is correlation ergodic.

Problem 25. a). If the WSS process $\{X(t)\}$ is given by $X(t) = \cos(\omega t + \varphi)$ where φ is uniformly distributed over $(-\pi, \pi)$ prove that $\{X(t)\}$ is correlation ergodic.

Solution:

To Prove $\{X(t)\}$ is correlation ergodic it is enough to show that when

$$\underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} X(t)X(t+\tau)\,dt = R_{XX}(\tau)$$

$$R_{XX}(\tau) = E\left[X(t)\,X(t+\tau)\right]$$

$$= E\left[\cos(\omega t + \theta)\cos(\omega t + \omega\tau + \theta)\right]$$

$$= E\left[\frac{\cos(2\omega t + \omega\tau + 2\theta) + \cos(\omega\tau)}{2}\right]$$

$$= \frac{1}{2}\int_{-\pi}^{\pi}\frac{1}{2\pi}\left[\cos(2\omega t + \omega\tau - 2\theta) + \cos\omega\tau\right]d\theta$$

$$= \frac{1}{4\pi}\left[\frac{\sin(2\omega t + \omega\tau + 2\theta)}{2} + \theta\cos\omega\tau\right]_{-\pi}^{\pi}$$

$$R_{XX}(\tau) = \frac{1}{2}\cos(\omega\tau)$$

Consider,

$$\underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} X(t)\,X(t+\tau)\,dt$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} \cos(\omega t + \theta)\cos(\omega t + \omega\tau + \theta)\,dt$$

$$= \underset{T\to\infty}{Lt} \frac{1}{4T}\int_{-T}^{T}\left[\cos(2\omega t + \omega\tau + 2\theta) + \cos(\omega\tau)\right]dt$$

$$= \underset{T\to\infty}{Lt} \frac{1}{4T}\left[\frac{\sin(2\omega t + \omega\tau + 2\theta)}{2\omega} + t\cos(\omega\tau)\right]_{-T}^{T}$$

$$= \frac{1}{2}\cos(\omega\tau)$$

$$\lim_{T\to\infty}\frac{1}{2\pi}\int_{-T}^{T} X(t)\,X(t+\tau)\,dt = \frac{1}{2}\cos(\omega\tau) = R(\tau)$$

$\therefore \{X(t)\}$ is correlation ergodic.

b). A random process $\{X(t)\}$ defined as $\{X(t)\} = A\cos\omega t + B\sin\omega t$, where A & B are the random variables with $E(A) = E(B) = 0$ and $E(A^2) = E(B^2)$ & $E(AB) = 0$. Prove that the process is mean ergodic.

Solution:

To prove that the process is mean ergodic we have to shoe that the ensemble mean is same as the mean in the time sense.

Given $E(A) = E(B) = 0$------------------ 1

$E(A^2) = E(B^2) = k(say)$ & $E(AB) = 0$----- 2.

Ensemble mean is

$E[X(t)] = E[A\cos\omega t + B\sin\omega t]$

$= E(A)\cos\omega t + E(B)\sin\omega t = 0$ Using (1)

Time Average

$$\overline{X(t)} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} (A\cos\omega t + B\sin\omega t)\,dt$$

$$= \lim_{T\to\infty} \frac{1}{2T}\left[\frac{A\sin\omega t}{\omega} - \frac{B\cos\omega t}{\omega}\right]_{-T}^{T}$$

$$= \lim_{T\to\infty} \frac{1}{2T}\left[\left(\frac{A\sin\omega t}{\omega} - \frac{B\cos\omega t}{\omega}\right) - \left(\frac{A\sin\omega(-T)}{\omega} - \frac{B\cos\omega(-T)}{\omega}\right)\right]$$

$$= \lim_{T\to\infty} \frac{1}{2T}\left[\frac{A\sin\omega t}{\omega} - \frac{B\cos\omega t}{\omega} + \frac{A\sin\omega T}{\omega} + \frac{B\cos\omega T}{\omega}\right]$$

$$= \lim_{T\to\infty} \frac{1}{2T}\left[\frac{2A\sin\omega T}{\omega}\right]$$

$$= \frac{A}{\omega}\lim_{T\to\infty}\left[\frac{\sin\omega T}{T}\right] = 0.$$

The ensemble mean =Time Average

Hence the process is mean ergodic.

26.a). Prove that in a Gaussian process if the variables are uncorrelated, then they are independent

Solution:

Let us consider the case of two variables X_{t_1} & X_{t_2}

If the are uncomelated then,

$r_{12} = r_{21} = r = 0$

Consequently the variance covariance matrix

$$\sum = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}$$

Matrix of co-factors $= \sum_{ij} = \begin{pmatrix} \sigma_2^2 & 0 \\ 0 & \sigma_1^2 \end{pmatrix}$

$\therefore |\sum| = \sigma_1^2\sigma_2^2 - 0 = \sigma_1^2\sigma_2^2$

Now $(X-\mu)\sum^{-1}(X-\mu)^1 = (X-\mu)\frac{E_{ij}}{|\sum|}(X-\mu)^1$

Let us consider

$(X-\mu)E_{ij}(X-\mu)^1$

$[X_1-\mu_1, X_2-\mu_2]\begin{bmatrix} \sigma_2^2 & 0 \\ 0 & \sigma_1^2 \end{bmatrix}\begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix} = \left[(x_1-\mu_1)\sigma_2^2 + 0 + 0 + (x_2-\mu_2)\sigma_1^2\right]\begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix}$

$= (x_1-\mu_1)^2\sigma_2^2 + (x_2-\mu_2)^2\sigma_1^2$

Now the joint density of X_{t_1} & X_{t_2} is

$$f(x_1,x_2) = \frac{1}{(2\pi)^{2/2}\sqrt{\sigma_1^2\sigma_2^2}}\, e^{-\frac{1}{2\sigma_1^2\sigma_2^2}\left[(x_1-\mu_1)^2\sigma_2^2 + (x_2-\mu_2)^2\sigma_1^2\right]}$$

$$= \frac{1}{(2\pi)\sigma_1\sigma_2}\, e^{-\frac{1}{2}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]}$$

$$= \frac{1}{2\pi\sigma_1\sigma_2}\, e^{-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2} e^{-\frac{1}{2}\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2}$$

$$= \frac{1}{\sigma_1\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2} \frac{1}{\sigma_2\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2}$$

$f(x_1,x_2) = f(x_1)\, f(x_2)$

$\therefore X_{t_1}$ & X_{t_2} are independent.

b). If $\{X(t)\}$ is a Gaussian process with $\mu(t) = 10$ & $C(t_1,t_2) = 16\, e^{-|t_1-t_2|}$. Find the probability that (i) $X(10) \le 8$ and (ii) $|X(10) - X(6)| \le 4$.

Solution:

Since $\{X(t)\}$ is a Gaussion process, any member of the process is a normal random variable.

$\therefore X(10)$ is a normal RV with mean $\mu(10) = 10$ and variance $C(10,10) = 16$.

$$P\{X(10) \le 8\} = P\left\{\frac{X(10)-10}{4} \le -0.5\right\}$$

$= P\{Z \le -0.5\}$ (Where Z is the standard normal RV)

$= 0.5 - P\{Z \le 0.5\}$

$= 0.5 - 0.1915 = 0.3085$.

(from normal tables)

$X(10) - X(6)$ is also a normal R V With mean $\mu(10) - \mu(6) = 10 - 10 = 0$

$$Var\{X(10) - X(6)\} = Var\{X(10)\} + Var\{X(6)\} - 2Cov\{X(10), X(6)\}$$
$$= Cov(10,10) + Cov(6,6) - 2Cov(10,6)$$
$$= 16 + 16 - 2 \times 16e^{-4} = 31.4139$$

$$P\{|X(10) - X(6)| \leq 4\} = P\left\{\left|\frac{X(10) - X(6)}{5.6048}\right| \leq \frac{4}{5.6048}\right\}$$
$$= P\{|Z| \leq 0.7137\}$$
$$= 2 \times 0.2611 = 0.5222.$$

Problem 27. a). Define a Markov chain. Explain how you would clarify the states and identify different classes of a Markov chain. Give example to each class.

Solution:

Markov Chain: If for all n,

$P\{X_n = a_n / X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2}, ..., X_0 = a_0\} = P\{X_n = a_n / X_{n-1} = a_{n-1}\}$ then the process $\{X_n\}$, $n = 0,1,2,...$ is called a Markov Chain.

<u>Classification of states of a Markov chain</u>

Irreducible: A Markov chain is said to be irreducible if every state can be reacted from every other state, where $P_{ij}^{(n)} > 0$ for some n and for all i & j.

Example: $\begin{bmatrix} 0.3 & 0.7 & 0 \\ 0.1 & 0.4 & 0.5 \\ 0 & 0.2 & 0.8 \end{bmatrix}$

Period: The Period d_i of a return state i is defined as the greatest common division of all m such that $P_{ij}^{(m)} > 0$

i.e., $d_i = GCD\{m : p_{ij}^{(m)} > 0\}$

State i is said to be periodic with period d_i if $d_i > 1$ and a periodic if $d_i = 1$.

Example:

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ So states are with period 2.

$\begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ The states are aperiodic as period of each state is 1.

Ergodic: A non null persistent and aperiodic state is called ergodic.

Example:

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{8} & \frac{1}{8} & \frac{1}{4} \end{bmatrix}$$ Here all the states are ergodic.

b). The one-step T.P.M of a Markov chain $\{X_n;\ n = 0,1,2,...\}$ having state space $S = \{1,2,3\}$ is $P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$ and the initial distribution is $\pi_0 = (0.7,\ 0.2,\ 0.1)$.

Find (i) $P(X_2 = 3/X_0 = 1)$ (ii) $P(X_2 = 3)$ (iii) $P(X_3 = 2,\ X_2 = 3,\ X_1 = 3,\ X_0 = 1)$.

Solution:

$$(i)\ P(X_2 = 3/X_0 = 1) = P(X_2 = 3/X_1 = 3)P(X_1 = 3/X_0 = 1) + P(X_3 = 3/X_1 = 2)P(X_1 = 2/X_0 = 1)$$
$$+P(X_2 = 3/X_1 = 1)P(X_1 = 1/X_0 = 1)$$
$$= (0.3)(0.4) + (0.2)(0.5) + (0.4)(0.1) = 0.26$$

$$P^2 = P.P = \begin{pmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{pmatrix}$$

(ii). $P(X_2 = 3) = \sum_{i=1}^{3} P(X_2 = 3/X_0 = i)P(X_0 = i)$

$$= P(X_2 = 3/X_0 = 1)P(X_0 = 1) + P(X_2 = 3/X_0 = 2)P(X_0 = 2)$$
$$+ P(X_2 = 3/X_0 = 3)P(X_0 = 3)$$
$$= P^2_{13}P(X_0 = 1) + P^2_{23}P(X_0 = 2) + P^2_{33}P(X_0 = 3)$$
$$= 0.26 \times 0.7 + 0.34 \times 0.2 + 0.29 \times 0.1 = 0.279$$

(iii). $P(X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 1)$

$$= P[X_0 = 1, X_1 = 3, X_2 = 3]P[X_3 = 2/X_0 = 1, X_1 = 3, X_2 = 3]$$
$$= P[X_0 = 1, X_1 = 3, X_2 = 3]P[X_3 = 2/X_2 = 3]$$
$$= P[X_0 = 1, X_1 = 3]P[X_2 = 3/X_0 = 1, X_1 = 3]P[X_3 = 2/X_2 = 3]$$
$$= P[X_0 = 1, X_1 = 3]P[X_2 = 3/X_1 = 3]P[X_3 = 2/X_2 = 3]$$
$$= P[X_0 = 1]P[X_1 = 3/X_0 = 1]P[X_2 = 3/X_1 = 3]P[X_3 = 2/X_2 = 3]$$
$$= (0.4)(0.3)(0.4)(0.7) = 0.0336$$

Problem 28. a). Let $\{X_n;\ n = 1,2,3,.....\}$ be a Markov chain with state space $S = \{0,1,2\}$ and 1 – step Transition probability matrix $P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 \end{bmatrix}$ (i) Is the chain ergodic? Explain (ii) Find the invariant probabilities.

Solution:

$$P^2 = P.P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{8} & \frac{3}{4} & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

$$P^3 = P^2P = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{8} & \frac{3}{4} & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{8} & \frac{3}{4} & \frac{1}{8} \\ \frac{3}{16} & \frac{5}{8} & \frac{3}{16} \\ \frac{1}{8} & \frac{3}{4} & \frac{1}{8} \end{pmatrix}$$

$P_{11}^{(3)} > 0,\ P_{13}^{(2)} > 0,\ P_{21}^{(2)} > 0,\ P_{22}^{(2)} > 0,\ P_{33}^{(2)} > 0$ and all other $P_{ij}^{(1)} > 0$

Therefore the chain is irreducible as the states are periodic with period 1

i.e., aperiodic since the chain is finite and irreducible, all are non null persistent

∴ The states are ergodic.

$$\begin{bmatrix} \pi_0 & \pi_1 & \pi_2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \pi_0 & \pi_1 & \pi_2 \end{bmatrix}$$

$\frac{\pi_1}{4} = \pi_0$ – – – – – – – – – – – – –(1)

$\pi_0 + \frac{\pi_1}{2} + \pi_2 = \pi_1$ – – – – – – – – – – – – –(2)

$\frac{\pi_1}{4} = \pi_2$ – – – – – – – – – – – – –(3)

$\pi_0 + \pi_1 + \pi_2 = 1$ – – – – – – – – – – – – –(4)

From (2) $\pi_0 + \pi_2 = \pi_1 - \frac{\pi_1}{2} = \frac{\pi_1}{2}$

$\therefore \pi_0 + \pi_1 + \pi_2 = 1$

$\frac{\pi_1}{2} + \pi_1 = 1$

$$\frac{3\pi_1}{2} = 1$$

$$\pi_1 = \frac{2}{3}$$

From (3) $\frac{\pi_1}{4} = \pi_2$

$$\pi_2 = \frac{1}{6}$$

Using (4) $\pi_0 + \frac{2}{3} + \frac{1}{6} = 1$

$$\pi_0 + \frac{4+1}{6} = 1$$

$$\pi_0 + \frac{5}{6} = 1 \Rightarrow \pi_0 = \frac{1}{6}$$

$$\therefore \pi_0 + \frac{1}{6}, \pi_1 = \frac{2}{3} \;\&\; \pi_2 = \frac{1}{6}.$$

b). Find the nature of the states of the Markov chain with the TPM $P = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix}$ and the state space $(1,2,3)$.

Solution:

$$P^2 = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

$$P^3 = P^2.P = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix} = P$$

$$P^4 = P^2.P^2 = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} = P^2$$

$$\therefore P^{2n} = P^2 \quad \& \quad P^{2n+1} = P$$

Also $P_{00}^{2} > 0,\ P_{01}^{1} > 0,\ P_{02}^{2} > 0$

$P_{10}^{1} > 0,\ P_{11}^{2} > 0,\ P_{12}^{1} > 0$

$P_{20}^{2} > 0,\ P_{21}^{1} > 0,\ P_{22}^{2} > 0$

∴ The Markov chain is irreducible

Also $P_{ii}^{2} = P_{ii}^{4} = ... > 0$ for all i

∴ The states of the chain have period 2. Since the chain is finite irreducible, all states are non null persistent. All states are not ergodic.

Problem 29. a). Three boys A, B and C are throwing a ball to each other. A always throws to B and B always throws to C, but C is as likely to throw the ball to B as to A. Find the TPM and classify the states.

Solution:

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{pmatrix} A & B & C \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

$$P^2 = P \times P = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$P^3 = P^2 \times P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

For any $i = 2,3$

$P_{ii}^{2} P_{ii}^{3},... > 0$

∴ G.C.D of $2,3,5,... = 1$

∴ The period of 2 and 3 is 1. The state with period 1 is aperiodic all states are ergodic.

b). A man either drives a car or catches a train to go to office each day. He never goes 2 days in a row by train but he drives one day, then the next day he is just as likely to drive again as he is to travel by train. Now suppose that one the first day of the week, the man tossed a fair dice and drove to work iff a 6 appeared. Find the probability that he takes a train on the third day and also the probability that on the third day and also the probability that he drives to work in the long run.

Solution:

State Space = (train, car)

The TPM of the chain is

$$P = \begin{matrix} & \\ T \\ C \end{matrix}\begin{pmatrix} T & C \\ 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

P (traveling by car) = P (getting 6 in the toss of the die)= $\frac{1}{6}$

& P (traveling by train) = $\frac{5}{6}$

$$P^{(1)} = \left(\frac{5}{6}, \frac{1}{6}\right)$$

$$P^{(2)} = P^{(1)}P = \left(\frac{5}{6}, \frac{1}{6}\right)\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \left(\frac{1}{12}, \frac{11}{12}\right)$$

$$P^{(3)} = P^{(2)}P = \left(\frac{1}{12}, \frac{11}{12}\right)\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \left(\frac{11}{24}, \frac{13}{24}\right)$$

P (the man travels by train on the third day) = $\frac{11}{24}$

Let $\pi = (\pi_1, \pi_2)$ be the limiting form of the state probability distribution or stationary state distribution of the Markov chain.

By the property of π, $\pi P = \pi$

$$(\pi_1 \;\; \pi_2)\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = (\pi_1 \;\; \pi_2)$$

$\frac{1}{2}\pi_2 = \pi_1$

$\pi_1 + \frac{1}{2}\pi_2 = \pi_2$

& $\pi_1 + \pi_2 = 1$

Solving $\pi_1 = \frac{1}{3}$ & $\pi_2 = \frac{2}{3}$

P{The man travels by car in the long run }= $\frac{2}{3}$.

Problem 30 a). Three are 2 white marbles in urn A and 3 red marbles in urn B. At each step of the process, a marble is selected from each urn and the 2 marbles selected are inter changed. Let the state a_i of the system be the number of red marbles in A after i changes. What is the probability that there are 2 red marbles in A after 3 steps? In the long run, what is the probability that there are 2 red marbles in urn A?

Solution:

State Space $\{X_n\} = (0,1,2)$ Since the number of ball in the urn A is always 2.

$$P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix} \end{matrix}$$

$X_n = 0$, $A = 2W$ (Marbles) $B = 3R$ (Marbles)

$X_{n+1} = 0 \qquad P_{00} = 0$

$X_{n+1} = 1 \qquad P_{01} = 1$

$X_{n+1} = 2 \qquad P_{02} = 0$

$X_n = 0$, $A = 1W \;\&\; 1R$ (Marbles) $B = 2R \;\&\; 1W$ (Marbles)

$X_{n+1} = 0 \qquad P_{10} = \frac{1}{6}$

$X_{n+1} = 1 \qquad P_{11} = \frac{1}{2}$

$X_{n+1} = 2 \qquad P_{12} = \frac{1}{3}$

$X_n = 2$, $A = 2R$ (Marbles) $B = 1R \;\&\; 2W$ (Marbles)

$X_{n+1} = 0 \qquad P_{20} = 0$

$X_{n+1} = 1 \qquad P_{21} = \frac{2}{3}$

$X_{n+1} = 2 \qquad P_{22} = \frac{1}{3}$

$P^{(0)} = (1,0,0)$ as there is not red marble in A in the beginning.

$P^{(1)} = P^{(0)}P = (0,\ 1,\ 0)$

$P^{(2)} = P^{(1)}P = \left(\frac{1}{6}, \frac{1}{2}, \frac{1}{3}\right)$

$P^{(3)} = P^{(2)}P = \left(\frac{1}{12}, \frac{23}{36}, \frac{5}{18}\right)$

∴ P (There are 2 red marbles in A after 3 steps) $= P\{X_3 = 2\} = P_2^{(3)} = \frac{5}{18}$

Let the stationary probability distribution of the chain be $\pi = (\pi_0, \pi_1, \pi_2)$.

By the property of π , $\pi P = \pi$ & $\pi_0 + \pi_1 + \pi_2 = 1$

$$\begin{pmatrix}\pi_0 & \pi_1 & \pi_2\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3}\\ 0 & \frac{2}{3} & \frac{1}{3}\end{pmatrix} = \begin{pmatrix}\pi_0 & \pi_1 & \pi_2\end{pmatrix}$$

$\frac{1}{6}\pi_1 = \pi_0$

$\pi_0 + \frac{1}{2}\pi_1 + \frac{2}{3}\pi_2 = \pi_1$

$\frac{1}{3}\pi_1 + \frac{1}{3}\pi_2 = \pi_2$

& $\pi_0 + \pi_1 + \pi_2 = 1$

Solving $\pi_0 = \frac{1}{10}, \pi_1 = \frac{6}{10}, \pi_2 = \frac{3}{10}$

P {here are 2 red marbles in A in the long run} = 0.3.

b). A raining process is considered as a two state Markov chain. If it rains the state is 0 and if it does not rain the state is 1. The TPM is $P = \begin{pmatrix}0.6 & 0.4\\ 0.2 & 0.8\end{pmatrix}$. If the Initial distribution is $(0.4, 0.6)$. Find it chance that it will rain on third day assuming that it is raining today.

Solution:

$$P^2 = \begin{pmatrix}0.6 & 0.4\\ 0.2 & 0.8\end{pmatrix}\begin{pmatrix}0.6 & 0.4\\ 0.2 & 0.8\end{pmatrix}$$

$$= \begin{pmatrix}0.44 & 0.56\\ 0.28 & 0.32\end{pmatrix}$$

P[rains on third day / it rains today $= P\left[X_3 = 0 / X_1 = 0\right]$

$= P_{00}^2 = 0.44$

UNIT – II TWO DIMENSIONAL RANDOM VARIABLES

Part.A

Problem 1. Let X and Y have joint density function $f(x,y) = 2,\ 0 < x < y < 1$. Find the marginal density function. Find the conditional density function Y given $X = x$.

Solution:

Marginal density function of X is given by

$$f_X(x) = f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_x^1 f(x,y)\,dy = \int_x^1 2dy = 2(y)_x^1$$

$$= 2(1-x),\ 0 < x < 1.$$

Marginal density function of Y is given by

$$f_Y(y) = f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_0^y 2dx = 2y, 0 < y < 1.$$

Conditional distribution function of Y given $X = x$ is $f\left(y/x\right) = \dfrac{f(x,y)}{f(x)} = \dfrac{2}{2(1-x)} = \dfrac{1}{1-x}$.

Problem 2. Verify that the following is a distribution function. $F(x) = \begin{cases} 0 & , x < -a \\ \frac{1}{2}\left(\frac{x}{a}+1\right), & -a < x < a \\ 1 & , x > a \end{cases}$.

Solution:

$F(x)$ is a distribution function only if $f(x)$ is a density function.

$$f(x) = \frac{d}{dx}\left[F(x)\right] = \frac{1}{2a}, \quad -a < x < a$$

$$\int_{-\infty}^{\infty} f(x) = 1$$

$$\therefore \int_{-a}^{a} \frac{1}{2a}dx = \frac{1}{2a}\left[x\right]_{-a}^{a} = \frac{1}{2a}\left[a-(-a)\right]$$

$$= \frac{1}{2a}.2a = 1.$$

Therefore, it is a distribution function.

Problem 3. Prove that $\int_{x_1}^{x_2} f_X(x)\,dx = p(x_1 < x < x_2)$

Solution:

$$\int_{x_1}^{x_2} f_X(x)\,dx = \left[F_X(x)\right]_{x_1}^{x_2}$$

$$= F_X(x_2) - F_X(x_1)$$

$$= P[X \le x_2] - P[X \le x_1]$$

$$= P[x_1 \le X \le x_2]$$

Problem 4. A continuous random variable X has a probability density function $f(x) = 3x^2$, $0 \le x \le 1$. Find $'a'$ such that $P(X \le a) = P(X > a)$.

Solution:

Since $P(X \le a) = P(X > a)$, each must be equal to $\frac{1}{2}$ because the probability is always 1.

$$\therefore P(X \le a) = \frac{1}{2}$$

$$\Rightarrow \int_0^a f(x)\,dx = \frac{1}{2}$$

$$\int_0^a 3x^2 dx = \frac{1}{2} \Rightarrow 3\left[\frac{x^3}{3}\right]_0^a = a^3 = \frac{1}{2}.$$

$$\therefore a = \left(\frac{1}{2}\right)^{\frac{1}{3}}$$

Problem5. Suppose that the joint density function

$$f(x, y) = \begin{cases} Ae^{-x-y}, & 0 \le x \le y, \ 0 \le y \le \infty \\ 0 & , otherwise \end{cases}$$ Determine A.

Solution:

Since $f(x, y)$ is a joint density function

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\,dxdy = 1.$$

$$\Rightarrow \int_0^{\infty}\int_0^{y} Ae^{-x}e^{-y}dxdy = 1$$

$$\Rightarrow A\int_0^\infty e^{-y}\left(\frac{e^{-x}}{-1}\right)_0^y dy = 1$$

$$\Rightarrow A\int_0^\infty \left[e^{-y} - e^{-2y}\right] dy = 1$$

$$\Rightarrow A\left[\frac{e^{-y}}{-1} - \frac{e^{-2y}}{-2}\right]_0^\infty = 1$$

$$\Rightarrow A\left[\frac{1}{2}\right] = 1 \Rightarrow A = 2$$

Problem 6. Examine whether the variables X and Y are independent, whose joint density function is $f(x, y) = xe^{-x(y+1)},\ 0 < x, y < \infty$.

Solution:

The marginal probability function of X is

$$f_X(x) = f(x) = \int_{-\infty}^{\infty} f(x, y)\,dy = \int_0^\infty xe^{-x(y+1)}dy$$

$$= x\left[\frac{e^{-x(y+1)}}{-x}\right]_0^\infty = -\left[0 - e^{-x}\right] = e^{-x},$$

The marginal probability function of Y is

$$f_Y(y) = f(y) = \int_{-\infty}^{\infty} f(x, y)\,dx = \int_0^\infty xe^{-x(y+1)}dx$$

$$= x\left\{\left[\frac{e^{-x(y+1)}}{-(y+1)}\right]_0^\infty - \left[\frac{e^{x(y+1)}}{(y+1)^2}\right]\right\}_0^\infty$$

$$= \frac{1}{(y+1)^2}$$

Here $f(x).f(y) = e^{-x} \times \dfrac{1}{(1+y)^2} \neq f(x, y)$

$\therefore X$ and Y are not independent.

Problem 7. If X has an exponential distribution with parameter 1. Find the pdf of $y = \sqrt{x}$

Solution:

Since $y = \sqrt{x}$, $x = y^2$

Since X has an exponential distribution with parameter 1, the pdf of X is given by

$$f_X(x) = e^{-x},\ x > 0 \qquad \left[\because f(x) = \lambda e^{-\lambda x}, \lambda = 1\right]$$

$$\therefore f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$$

$$= e^{-x}2y = 2ye^{-y^2}$$

$f_Y(y) = 2ye^{-y^2}, \; y > 0$

Problem 8. If X is uniformly distributed random variable in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, Find the probability density function of $Y = tanX$.

Solution:

Given $Y = tanX \Rightarrow x = tan^{-1}y$

$\therefore \frac{dx}{dy} = \frac{1}{1+y^2}$

Since X is uniformly distribution in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$f_X(x) = \frac{1}{b-a} = \frac{1}{\frac{\pi}{2}+\frac{\pi}{2}}$

$f_X(x) = \frac{1}{\pi}, \; -\frac{\pi}{2} < x < \frac{\pi}{2}$

Now $f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = \frac{1}{\pi}\left(\frac{1}{1+y^2}\right), \; -\infty < y < \infty$

$\therefore f_Y(y) = \frac{1}{\pi(1+y^2)}, \; -\infty < y < \infty$

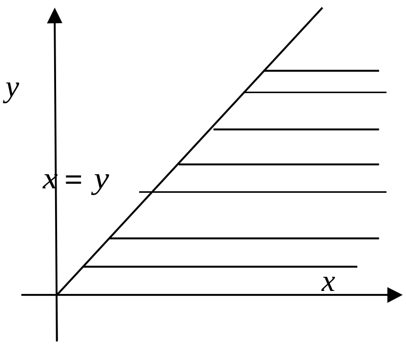
Problem 9. If the Joint probability density function of (x, y) is given by $f(x, y) = 24y(1-x)$, $0 \le y \le x \le 1$ Find $E(XY)$.

Solution:

$E(xy) = \int_0^1\int_y^1 xyf(x, y)\,dxdy$

$= 24\int_0^1\int_y^1 xy^2(1-x)\,dxdy$

$= 24\int_0^1 y^2\left[\frac{1}{6} - \frac{y^2}{2} + \frac{y^3}{3}\right]dy = \frac{4}{15}.$



Problem 10. If X and Y are random Variables, Prove that $Cov(X,Y) = E(XY) - E(X)E(Y)$

Solution:

$cov(X,Y) = E\left[\left(X - E(X)\right)\left(Y - E(Y)\right)\right]$

$= E\left(XY - \overline{X}Y - \overline{Y}X + \overline{X}\overline{Y}\right)$

$= E(XY) - \overline{X}E(Y) - \overline{Y}E(X) + \overline{X}\overline{Y}$

$= E(XY) - \overline{X}\overline{Y} - \overline{X}\overline{Y} + \overline{X}\overline{Y}$

$$= E(XY) - E(X)E(Y) \qquad \left[\because E(X) = \bar{X}, E(Y) = \bar{Y}\right]$$

Problem 11. If X and Y are independent random variables prove that $cov(x, y) = 0$

Proof:

$cov(x, y) = E(xy) - E(x)E(y)$

But if X and Y are independent then $E(xy) = E(x)E(y)$

$cov(x, y) = E(x)E(y) - E(x)E(y)$

$cov(x, y) = 0.$

Problem 12. Write any two properties of regression coefficients.

Solution:

1. Correction coefficients is the geometric mean of regression coefficients
2. If one of the regression coefficients is greater than unity then the other should be less than 1.

$$b_{xy} = r\frac{\sigma_y}{\sigma_x} \text{ and } b_{yx} = r\frac{\sigma_x}{\sigma_y}$$

If $b_{xy} > 1$ then $b_{yx} < 1$.

Problem 13. Write the angle between the regression lines.

Solution: The slopes of the regression lines are

$$m_1 = r\frac{\sigma_y}{\sigma_x}, m_2 = \frac{1}{r}\frac{\sigma_y}{\sigma_x}$$

If θ is the angle between the lines, Then

$$\tan\theta = \frac{\sigma_x\sigma_y}{\sigma_x^2 + \sigma_y^2}\left[\frac{1 - r^2}{r}\right]$$

When $r = 0$, that is when there is no correlation between x and y, $\tan\theta = \infty$ (or) $\theta = \frac{\pi}{2}$ and so the regression lines are perpendicular

When $r = 1$ or $r = -1$, that is when there is a perfect correlation $+ve$ or $-ve$, $\theta = 0$ and so the lines coincide.

Problem 14. State central limit theorem

Solution:

If $X_1, X_2....X_n$ is a sequence of independent random variable $E(X_i) = \mu_i$ and $Var(X_i) = \sigma_i^2, i = 1, 2,.....n$ and if $S_n = X_1 + X_2 + ... + X_n$ then under several conditions S_n follows a normal distribution with mean $\mu = \sum_{i=1}^{n}\mu_i$ and variance $\sigma^2 = \sum_{i=1}^{n}\sigma_i^2$ as $n \to \infty$.

Problem 15. i). Two random variables are said to be orthogonal if correlation is zero.

ii). If $X = Y$ then correlation coefficient between them is <u>1.</u>

Part-B

Problem 16. a). The joint probability density function of a bivariate random variable (X,Y) is $f_{XY}(x,y) = \begin{cases} k(x+y), & 0 < x < 2,\ 0 < y < 2 \\ 0 & , otherwise \end{cases}$ where $'k'$ is a constant.

i. Find k.
ii. Find the marginal density function of X and Y.
iii. Are X and Y independent?
iv. Find $f_{Y/X}\left(y/x\right)$ and $f_{X/Y}\left(x/y\right)$.

Solution:

(i). Given the joint probability density function of a brivate random variable (X,Y) is

$$f_{XY}(x,y) = \begin{cases} K(x+y), & 0 < x < 2,\ 0 < y < 2 \\ 0 & , otherwise \end{cases}$$

Here $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dxdy = 1 \Rightarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(x+y)\,dxdy = 1$

$$\int_0^2\int_0^2 K(x+y)\,dxdy = 1 \Rightarrow K\int_0^2\left[\frac{x^2}{2} + xy\right]_0^2 dy = 1$$

$$\Rightarrow K\int_0^2 (2+2y)\,dy = 1$$

$$\Rightarrow K\left[2y + y^2\right]_0^2 = 1$$

$$\Rightarrow K[8-0] = 1$$

$$\Rightarrow K = \frac{1}{8}$$

(ii). The marginal p.d.f of X is given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \frac{1}{8}\int_0^2 (x+y)\,dy$$

$$= \frac{1}{8}\left[xy + \frac{y^2}{2}\right]_0^2 = \frac{1+x}{4}$$

$\therefore$ The marginal p.d.f of X is

$$f_X(x) = \begin{cases} \frac{x+1}{4}, & 0 < x < 2 \\ 0 & , otherwise \end{cases}$$

The marginal p.d.f of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \frac{1}{8}\int_0^2 (x+y)\,dx$$

$$= \frac{1}{8}\left[\frac{x^2}{2} + yx\right]_0^2$$

$$= \frac{1}{8}\left[2 + 2y\right] = \frac{y+1}{4}$$

$\therefore$The marginal p.d.f of Y is

$$f_Y(y) = \begin{cases} \frac{y+1}{4}, & 0 < y < 2 \\ 0, & otherwise \end{cases}$$

(iii). To check whether X and Y are independent or not.

$$f_X(x)\, f_Y(y) = \frac{(x+1)}{4}\frac{(y+1)}{4} \neq f_{XY}(x, y)$$

Hence X and Y are not independent.

(iv). Conditional p.d.f $f_{Y/X}\left(y/x\right)$ is given by

$$f_{Y/X}\left(y/x\right) = \frac{f(x, y)}{f_X(x)} = \frac{\frac{1}{8}(x+y)}{\frac{1}{4}(x+1)} = \frac{1}{2}\frac{(x+y)}{(x+1)}$$

$$f_{Y/X}\left(y/x\right) = \frac{1}{2}\left(\frac{x+y}{x+1}\right), 0 < x < 2,\ 0 < y < 2$$

(v) $$P\left(0 < y < \tfrac{1}{2} \Big/ x = 1\right) = \int_0^{\frac{1}{2}} f_{Y/X}\left(y/x = 1\right) dy$$

$$= \frac{1}{2}\int_0^{\frac{1}{2}} \frac{1+y}{2} dy = \frac{5}{32}.$$

Problem 17.a). If X and Y are two random variables having joint probability density function

$$f(x, y) = \begin{cases} \frac{1}{8}(6 - x - y), & 0 < x < 2,\ 2 < y < 4 \\ 0, & otherwise \end{cases}$$

Find (i) $P(X < 1 \cap Y < 3)$ (ii) $P(X + Y < 3)$ (iii) $P\left(X < 1 / Y < 3\right)$.

b). Three balls are drawn at random without replacement from a box containing 2 white, 3 red and 4 black balls. If X denotes the number of white balls drawn and Y denotes the number of red balls drawn find the joint probability distribution of (X, Y).

Solution:

a).

$$P(X < 1 \cap Y < 3) = \int_{y=-\infty}^{y=3}\int_{x=-\infty}^{x=1} f(x, y)\, dxdy$$

$$= \int_{y=2}^{y=3}\int_{x=0}^{x=1} \frac{1}{8}(6-x-y)\,dxdy$$

$$= \frac{1}{8}\int_2^3\int_0^1 (6-x-y)\,dxdy$$

$$= \frac{1}{8}\int_2^3 \left[6x - \frac{x^2}{2} - xy\right]_0^1 dy$$

$$= \frac{1}{8}\int_2^3 \left[\frac{11}{2} - y\right] dy = \frac{1}{8}\left[\frac{11y}{2} - \frac{y^2}{2}\right]_2^3$$

$$P(X<1 \cap Y<3) = \frac{3}{8}$$

(ii). $P(X+Y<3) = \int_0^1\int_2^{3-x} \frac{1}{8}(6-x-y)\,dydx$

$$= \frac{1}{8}\int_0^1 \left[6y - xy - \frac{y^2}{2}\right]_2^{3-x} dx$$

$$= \frac{1}{8}\int_0^1 \left[6(3-x) - x(3-x) - \frac{(3-x)^2}{2} - [12-2x-2]\right] dx$$

$$= \frac{1}{8}\int_0^1 \left[18 - 6x - 3x + x^2 - \frac{(9+x^2-6x)}{2} - (10-2x)\right] dx$$

$$= \frac{1}{8}\int_0^1 \left[18 - 9x + x^2 - \frac{9}{2} - \frac{x^2}{2} + \frac{6x}{2} - 10 + 2x\right] dx$$

$$= \frac{1}{8}\int_0^1 \left[\frac{7}{2} - 4x + \frac{x^2}{2}\right] dx$$

$$= \frac{1}{8}\left[\frac{7x}{2} - \frac{4x^2}{2} + \frac{x^3}{6}\right]_0^1 = \frac{1}{8}\left[\frac{7}{2} - 2 + \frac{1}{6}\right]$$

$$= \frac{1}{8}\left[\frac{21-12+1}{6}\right] = \frac{1}{8}\left(\frac{10}{6}\right) = \frac{5}{24}.$$

(iii). $P\left(X<1 \middle/ Y<3\right) = \dfrac{P(x<1 \cap y<3)}{P(y<3)}$

The Marginal density function of Y is $f_Y(y) = \int_0^2 f(x,y)\,dx$

$$= \int_0^2 \frac{1}{8}(6-x-y)\,dx$$

$$= \frac{1}{8}\left[6x - \frac{x^2}{2} - yx\right]_0^2$$

$$= \frac{1}{8}\left[12 - 2 - 2y\right]$$

$$= \frac{5-y}{4},\ 2 < y < 4.$$

$$P\left(X < 1 \big/ Y < 3\right) = \frac{\int_{x=0}^{x=1}\int_{y=2}^{y=3} \frac{1}{8}\left(6 - x - y\right) dxdy}{\int_{y=2}^{y=3} f_Y\left(y\right) dy}$$

$$= \frac{\frac{3}{8}}{\int_2^3 \left(\frac{5-y}{4}\right) dy} = \frac{\frac{3}{8}}{\frac{1}{4}\left[5y - \frac{y^2}{2}\right]_2^3}$$

$$= \frac{3}{8} \times \frac{8}{5} = \frac{3}{5}.$$

b). Let X takes 0, 1, 2 and Y takes 0, 1, 2 and 3.

$P\left(X = 0, Y = 0\right) = P($drawing 3 balls none of which is white or red)

$= P($ all the 3 balls drawn are black)

$$= \frac{4C_3}{9C_3} = \frac{4 \times 3 \times 2 \times 1}{9 \times 8 \times 7} = \frac{1}{21}.$$

$P\left(X = 0, Y = 1\right) = P($drawing 1 red ball and 2 black balls)

$$= \frac{3C_1 \times 4C_2}{9C_3} = \frac{3}{14}$$

$P\left(X = 0, Y = 2\right) = P($ drawing 2 red balls and 1 black ball)

$$= \frac{3C_2 \times 4C_1}{9C_3} = \frac{3 \times 2 \times 4 \times 3}{9 \times 8 \times 7} = \frac{1}{7}.$$

$P\left(X = 0, Y = 3\right) = P($ all the three balls drawn are red and no white ball)

$$= \frac{3C_3}{9C_3} = \frac{1}{84}$$

$P\left(X = 1, Y = 0\right) = P($drawing 1White and no red ball)

$$= \frac{2C_1 \times 4C_2}{9C_3} = \frac{\frac{2 \times 4 \times 3}{1 \times 2}}{\frac{9 \times 8 \times 7}{1 \times 2 \times 3}}$$

$$= \frac{12 \times 1 \times 2 \times 3}{9 \times 8 \times 7} = \frac{1}{7}$$

$P(X = 1, Y = 1) = P($ drawing 1White and 1 red ball)

$$= \frac{2C_1 \times 3C_1}{9C_3} = \frac{2 \times 3}{\frac{9 \times 8 \times 7}{1 \times 2 \times 3}} = \frac{2}{7}$$

$P(X = 1, Y = 2) = P($ drawing 1White and 2 red ball)

$$= \frac{2C_1 \times 3C_2}{9C_3} = \frac{2 \times 3 \times 2}{\frac{9 \times 8 \times 7}{1 \times 2 \times 3}} = \frac{1}{14}$$

$P(X = 1, Y = 3) = 0$ (Since only three balls are drawn)

$P(X = 2, Y = 0) = P($ drawing 2 white balls and no red balls)

$$= \frac{2C_2 \times 4C_1}{9C_3} = \frac{1}{21}$$

$P(X = 2, Y = 1) = P($ drawing 2 white balls and no red balls)

$$= \frac{2C_2 \times 3C_1}{9C_3} = \frac{1}{28}$$

$P(X = 2, Y = 2) = 0$

$P(X = 2, Y = 3) = 0$

The joint probability distribution of (X, Y) may be represented as

X \ Y	0	1	2	3
0	$\frac{1}{21}$	$\frac{3}{14}$	$\frac{1}{7}$	$\frac{1}{84}$
1	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{1}{14}$	0
2	$\frac{1}{21}$	$\frac{1}{28}$	0	0

Problem 18.a). Two fair dice are tossed simultaneously. Let X denotes the number on the first die and Y denotes the number on the second die. Find the following probabilities.

(i) $P(X + Y) = 8$, (ii) $P(X + Y \geq 8)$, (iii) $P(X = Y)$ and (iv) $P\left(X + Y = 6 \big/ Y = 4\right)$.

b) The joint probability mass function of a bivariate discrete random variable (X, Y) in given by the table.

Y \ X	1	2	3
1	0.1	0.1	0.2
2	0.2	0.3	0.1

Find

i. The marginal probability mass function of X and Y.
ii. The conditional distribution of X given $Y = 1$.
iii. $P\left(X + Y < 4\right)$

Solution:

a). Two fair dice are thrown simultaneously

$$S = \begin{Bmatrix} (1,1)(1,2)\ldots(1,6) \\ (2,1)(2,2)\ldots(2,6) \\ . \quad . \quad \ldots \quad . \\ (6,1)(6,2)\ldots(6,6) \end{Bmatrix}, \quad n(S) = 36$$

Let X denotes the number on the first die and Y denotes the number on the second die.

Joint probability density function of $\left(X,Y\right)$ is $P\left(X = x, Y = y\right) = \frac{1}{36}$ for

$x = 1,2,3,4,5,6$ and $y = 1,2,3,4,5,6$

(i) $X + Y = \{$ the events that the no is equal to 8 $\}$

$= \{(2,6),(3,5),(4,4),(5,3),(6,2)\}$

$$P\left(X + Y = 8\right) = P\left(X = 2, Y = 6\right) + P\left(X = 3, Y = 5\right) + P\left(X = 4, Y = 4\right)$$
$$+ P\left(X = 5, Y = 3\right) + P\left(X = 6, Y = 2\right)$$
$$= \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} = \frac{5}{36}$$

(ii) $P\left(X + Y \geq 8\right)$

$$X + Y = \begin{Bmatrix} (2,6) \\ (3,5),(3,6) \\ (4,4),(4,5),(4,6) \\ (5,3),(5,4)(5,5),(5,6) \\ (6,2),(6,3),(6,4),(6,5)(6,6) \end{Bmatrix}$$

$$\therefore P\left(X + Y \geq 8\right) = P\left(X + Y = 8\right) + P\left(X + Y = 9\right) + P\left(X + Y = 10\right)$$
$$+ P\left(X + Y = 11\right) + P\left(X + Y = 12\right)$$
$$= \frac{5}{36} + \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{15}{36} = \frac{5}{12}$$

(iii) $P\left(X = Y\right)$

$$P\left(X = Y\right) = P\left(X = 1, Y = 1\right) + P\left(X = 2, Y = 2\right) + \ldots\ldots + P\left(X = 6, Y = 6\right)$$
$$= \frac{1}{36} + \frac{1}{36} + \ldots\ldots\ldots + \frac{1}{36} = \frac{6}{36} = \frac{1}{6}$$

(iv) $P\left(X + Y = 6 \middle/ Y = 4\right) = \dfrac{P\left(X + Y = 6 \cap Y = 4\right)}{P\left(Y = 4\right)}$

Now $P\left(X+Y=6 \cap Y=4\right)=\dfrac{1}{36}$

$P\left(Y=4\right)=\dfrac{6}{36}$

$$\therefore P\left(X+Y=6 \big/ Y=4\right)=\frac{\frac{1}{36}}{\frac{6}{36}}=\frac{1}{6}.$$

b). The joint probability mass function of $\left(X,Y\right)$ is

Y \ X	1	2	3	Total
1	0.1	0.1	0.2	0.4
2	0.2	0.3	0.1	0.6
Total	0.3	0.4	0.3	1

From the definition of marginal probability function

$$P_X\left(x_i\right)=\sum_{y_j} P_{XY}\left(x_i,y_j\right)$$

When $X=1$,

$$P_X\left(x_i\right)=P_{XY}\left(1,1\right)+P_{XY}\left(1,2\right)$$
$$=0.1+0.2=0.3$$

When $X=2$,

$$P_X\left(x=2\right)=P_{XY}\left(2,1\right)+P_{XY}\left(2,2\right)$$
$$=0.2+0.3=0.4$$

When $X=3$,

$$P_X\left(x=3\right)=P_{XY}\left(3,1\right)+P_{XY}\left(3,2\right)$$
$$=0.2+0.1=0.3$$

$\therefore$ The marginal probability mass function of X is

$$P_X\left(x\right)=\begin{cases}0.3 & when\ \ x=1\\ 0.4 & when\ \ x=2\\ 0.3 & when\ \ x=3\end{cases}$$

The marginal probability mass function of Y is given by $P_Y\left(y_j\right)=\sum_{x_i} P_{XY}\left(x_i,y_j\right)$

When $Y=1$, $P_Y\left(y=1\right)=\sum_{x_i=1}^{3} P_{XY}\left(x_i,1\right)$

$$=P_{XY}\left(1,1\right)+P_{XY}\left(2,1\right)+P_{XY}\left(3,1\right)$$
$$=0.1+0.1+0.2=0.4$$

When $Y=2$, $P_Y\left(y=2\right)=\sum_{x_i=1}^{3} P_{XY}\left(x_i,2\right)$

$$=P_{XY}\left(1,2\right)+P_{XY}\left(2,2\right)+P_{XY}\left(3,2\right)$$
$$=0.2+0.3+0.1=0.6$$

$\therefore$ Marginal probability mass function of Y is

$$P_Y(y) = \begin{cases} 0.4 & when \;\; y = 1 \\ 0.6 & when \;\; y = 2 \end{cases}$$

(ii) The conditional distribution of X given $Y = 1$ is given by

$$P\left(X = x / Y = 1 \right) = \frac{P\left(X = x \cap Y = 1 \right)}{P\left(Y = 1 \right)}$$

From the probability mass function of Y, $P(y = 1) = P_y(1) = 0.4$

When $X = 1$, $P\left(X = 1 / Y = 1 \right) = \dfrac{P\left(X = 1 \cap Y = 1 \right)}{P\left(Y = 1 \right)}$

$$= \frac{P_{XY}(1,1)}{P_Y(1)} = \frac{0.1}{0.4} = 0.25$$

When $X = 2$, $P\left(X = 2 / Y = 1 \right) = \dfrac{P_{XY}(2,1)}{P_Y(1)} = \dfrac{0.1}{0.4} = 0.25$

When $X = 3$, $P\left(X = 3 / Y = 1 \right) = \dfrac{P_{XY}(3,1)}{P_Y(1)} = \dfrac{0.2}{0.4} = 0.5$

(iii). $P\left(X + Y < 4 \right) = P\left\{ (x, y) / x + y < 4 \;\; Where \;\; x = 1, 2, 3; \;\; y = 1, 2 \right\}$

$$= P\left\{ (1,1), (1,2), (2,1) \right\}$$

$$= P_{XY}(1,1) + P_{XY}(1,2) + P_{XY}(2,1)$$

$$= 0.1 + 0.1 + 0.2 = 0.4$$

Problem 19.a). If X and Y are two random variables having the joint density function $f(x, y) = \frac{1}{27}(x + 2y)$ where x and y can assume only integer values 0, 1 and 2, find the conditional distribution of Y for $X = x$.

b). The joint probability density function of (X, Y) is given by $f_{XY}(x, y) = \begin{cases} xy^2 + \dfrac{x^2}{8}, & 0 \le x \le 2, \;\; 0 \le y \le 1 \\ 0 & , \; otherwise \end{cases}$. Find (i) $P(X > 1)$, (ii) $P(X < Y)$ and (iii) $P(X + Y \le 1)$

Solution:

a). Given X and Y are two random variables having the joint density function

$$f(x, y) = \frac{1}{27}(x + 2y) - - - -(1)$$

Where $x = 0, 1, 2$ and $y = 0, 1, 2$

Then the joint probability distribution X and Y becomes as follows

X \ Y	0	1	2	$f_1(x)$
0	0	$\frac{1}{27}$	$\frac{2}{27}$	$\frac{3}{27}$
1	$\frac{2}{27}$	$\frac{3}{27}$	$\frac{4}{27}$	$\frac{9}{27}$
2	$\frac{4}{27}$	$\frac{5}{27}$	$\frac{6}{27}$	$\frac{15}{27}$

The marginal probability distribution of X is given by $f_1(X) = \sum_j P(x, y)$ and is calculated in the above column of above table.

The conditional distribution of Y for X is given by $f_1\left(Y = y / X = x\right) = \dfrac{f(x, y)}{f_1(x)}$ and is obtained in the following table.

Y \ X	0	1	2
0	0	$\frac{1}{3}$	$\frac{2}{3}$
1	$\frac{1}{9}$	$\frac{3}{9}$	$\frac{5}{9}$
2	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$

$$P\left(Y = 0 / X = 0\right) = \frac{P(X = 0, Y = 0)}{P(X = 0)} = \frac{0}{\frac{6}{27}} = 0$$

$$P\left(Y = 1 / X = 0\right) = \frac{P(X = 0, Y = 1)}{P(X = 0)} = \frac{\frac{2}{27}}{\frac{6}{27}} = \frac{1}{3}$$

$$P\left(Y = 2 / X = 0\right) = \frac{P(X = 0, Y = 2)}{P(X = 0)} = \frac{\frac{4}{27}}{\frac{6}{27}} = \frac{2}{3}$$

$$P\left(Y = 0 / X = 1\right) = \frac{P(X = 1, Y = 0)}{P(X = 1)} = \frac{\frac{1}{27}}{\frac{9}{27}} = \frac{1}{9}$$

$$P\left(Y = 1 / X = 1\right) = \frac{P(X = 1, Y = 1)}{P(X = 1)} = \frac{\frac{3}{27}}{\frac{9}{27}} = \frac{3}{9} = \frac{1}{3}$$

$$P\left(Y=2 \middle/ X=1\right)=\frac{P(X=1,Y=2)}{P(X=1)}=\frac{\frac{5}{27}}{\frac{9}{27}}=\frac{5}{9}$$

$$P\left(Y=0 \middle/ X=2\right)=\frac{P(X=2,Y=0)}{P(X=2)}=\frac{\frac{2}{27}}{\frac{12}{27}}=\frac{1}{6}$$

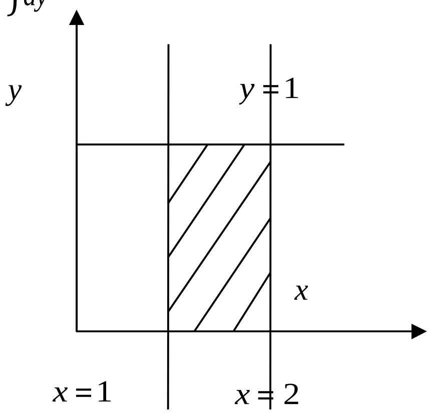
$$P\left(Y=1 \middle/ X=2\right)=\frac{P(X=2,Y=1)}{P(X=2)}=\frac{\frac{4}{27}}{\frac{12}{27}}=\frac{1}{3}$$

$$P\left(Y=2 \middle/ X=2\right)=\frac{P(X=2,Y=2)}{P(X=2)}=\frac{\frac{6}{27}}{\frac{12}{27}}=\frac{1}{2}$$

b). Given the joint probability density function of (X,Y) is $f_{XY}(x+y)=xy^2+\frac{x^2}{8}$, $0\le x\le 2,\ 0\le y\le 1$

(i). $P(X>1)=\int_1^{\infty} f_X(x)\,dx$

The Marginal density function of X is $f_X(x)=\int_0^1 f(x,y)\,dy$



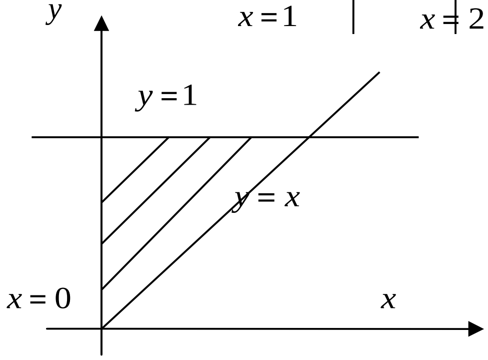
$$f_X(x)=\int_0^1\left(xy^2+\frac{x^2}{8}\right)dy$$

$$=\left[\frac{xy^2}{3}+\frac{x^2y}{8}\right]_0^1=\frac{x}{3}+\frac{x^2}{8},\ 1<x<2$$

$$P(X>1)=\int_1^2\left(\frac{x}{3}+\frac{x^2}{8}\right)dx$$

$$=\left[\frac{x^2}{6}+\frac{x^3}{24}\right]_1^2=\frac{19}{24}.$$

(ii) $P(X<Y)=\iint_{R_2} f_{XY}(x,y)\,dxdy$



$$P(X<Y)=\int_{y=0}^{1}\int_{x=0}^{y}\left(xy^2+\frac{x^2}{8}\right)dxdy$$

$$=\int_0^1\left[\frac{x^2y^2}{2}+\frac{x^3}{24}\right]_0^y dy$$

$$= \int_0^1 \left(\frac{y^4}{2} + \frac{y^3}{24} \right) dy = \left[\frac{y^5}{10} + \frac{y^4}{96} \right]_0^1$$

$$= \frac{1}{10} + \frac{1}{96} = \frac{96+10}{960} = \frac{53}{480}$$

(iii) $P\left(X+Y \le 1\right) = \iint_{R_3} f_{XY}\left(x,y\right) dxdy$

Where R_3 is the region

$$P\left(X+Y \le 1\right) = \int_{y=0}^{1} \int_{x=0}^{1-y} \left(xy^2 + \frac{x^2}{8} \right) dxdy$$

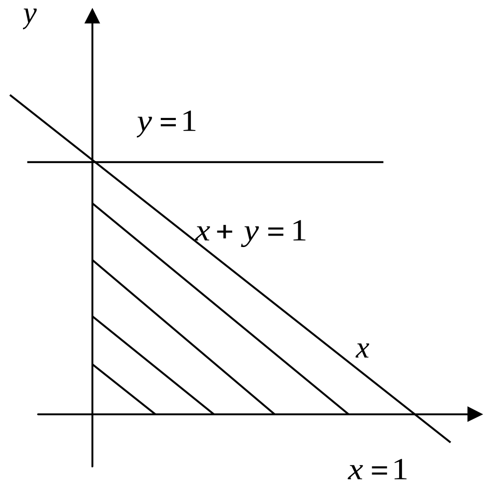
$$= \int_{y=0}^{1} \left[\frac{x^2 y^2}{2} + \frac{x^3}{24} \right]_0^{1-y} dy$$

$$= \int_{y=0}^{1} \left(\frac{(1-y)^2 y^2}{2} + \frac{(1-y)^3}{24} \right) dy$$

$$= \int_0^1 \left(\frac{\left(1+y^2-2y\right) y^2}{2} + \frac{(1-y)^3}{24} \right) dy$$

$$= \left[\left[\frac{y^3}{3} + \frac{y^5}{5} - \frac{2y^2}{4} \right] \frac{1}{2} + \frac{(1-y)^4}{96} \right]_0^1$$

$$= \frac{1}{6} + \frac{1}{10} - \frac{1}{4} + \frac{1}{96} = \frac{13}{480}.$$



Problem20 a). If the joint distribution functions of X and Y is given by

$$F\left(x,y\right) = \begin{cases} \left(1-e^{x}\right)\left(1-e^{-y}\right), & x>0,\ y>0 \\ 0 & , otherwise \end{cases}$$

i. Find the marginal density of X and Y.

ii. Are X and Y independent.

iii. $P\left(1<X<3,\ 1<Y<2\right)$.

b). The joint probability distribution of X and Y is given by

$$f\left(x,y\right) = \begin{cases} \frac{6-x-y}{8}, & 0<x<2,\ 2<y<4 \\ 0 & , otherwise \end{cases}$$

. Find $P\left(1<Y<3 \middle/ X=2\right)$.

Solution:

a). Given $F\left(x,y\right) = \left(1-e^{-x}\right)\left(1-e^{-y}\right)$

$$= 1 - e^{-x} - e^{-y} + e^{-(x+y)}$$

The joint probability density function is given by

$$f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y}$$
$$= \frac{\partial^2}{\partial x \partial y}\left[1 - e^{-x} - e^{-y} + e^{-(x+y)}\right]$$
$$= \frac{\partial}{\partial x}\left[e^{-y} - e^{-(x+y)}\right]$$
$$\therefore \quad f(x,y) = \begin{cases} e^{-(x+y)}, & x \ge 0,\ y \ge 0 \\ 0 & , \ otherwise \end{cases}$$

(ii) The marginal probability function of X is given by

$$f(x) = f_X(x)$$
$$= \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^{\infty} e^{-(x+y)}dy$$
$$= \left[\frac{e^{-(x+y)}}{-1}\right]_0^{\infty}$$
$$= \left[-e^{-(x+y)}\right]_0^{\infty}$$
$$= e^{-x},\ x > 0$$

The marginal probability function of Y is

$$f(y) = f_Y(y)$$
$$= \int_{-\infty}^{\infty} f(x,y)\,dx$$
$$= \int_0^{\infty} e^{-(x+y)}dx = \left[-e^{-(x+y)}\right]_0^{\infty}$$
$$= e^{-y},\ y > 0$$
$$\therefore f(x)\,f(y) = e^{-x}e^{-y} = e^{-(x+y)} = f(x,y)$$

$\therefore X$ and Y are independent.

(iii) $P(1 < X < 3, 1 < Y < 2) = P(1 < X < 3) \times P(1 < Y < 2)$ [Since X and Y are independent]

$$= \int_1^3 f(x)dx \times \int_1^2 f(y)\,dy.$$
$$= \int_1^3 e^{-x}dx \times \int_1^2 e^{-y}dy$$
$$= \left[\frac{e^{-x}}{-}\right]_1^3 \times \left[\frac{e^{-y}}{-}\right]_1^2$$
$$= \left(-e^{-3} + e^{-1}\right)\left(-e^{-2} + e^{-1}\right)$$
$$= e^{-5} - e^{-4} - e^{-3} + e^{-2}$$

b). $P\left(1 < Y < 3 \big/ X = 2\right) = \int_1^3 f\left(y \big/ x = 2\right) dy$

$$f_X(x) = \int_2^4 f(x,y)\,dy$$
$$= \int_2^4 \left(\frac{6 - x - y}{8}\right) dy$$
$$= \frac{1}{8}\left(6y - xy - \frac{y^2}{2}\right)_2^4$$
$$= \frac{1}{8}\left(16 - 4x - 10 + 2x\right)$$

$$f\left(y \big/ x\right) = \frac{f(x,y)}{f(x)} = \frac{\frac{6 - x - y}{8}}{\frac{6 - 2x}{8}} = \frac{6 - x - y}{6 - 2x}$$

$$P\left(1 < Y < 3 \big/ X = 2\right) = \int_1^3 f\left(y \big/ x = 2\right) dy$$
$$= \int_2^3 \left(\frac{4 - y}{2}\right) dy$$
$$= \frac{1}{2}\left[4y - \frac{y^2}{2}\right]_2^3$$
$$= \frac{1}{2}\left[4y - \frac{y^2}{2}\right]_2^3 = \frac{1}{2}\left[14 - \frac{17}{2}\right] = \frac{11}{4}.$$

Problem 21. a). Two random variables X and Y have the following joint probability density function $f(x,y) = \begin{cases} 2 - x - y, & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & , otherwise \end{cases}$. Find the marginal probability density function of X and Y. Also find the covariance between X and Y.

b). If $f(x,y) = \frac{6 - x - y}{8}, 0 \le x \le 2, 2 \le y \le 4$ for a bivariate (X,Y), find the correlation coefficient

Solution:

a) Given the joint probability density function $f(x,y) = \begin{cases} 2 - x - y, & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & , otherwise \end{cases}$

Marginal density function of X is $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

$$= \int_0^1 (2 - x - y)\,dy$$

$$= \left[2y - xy - \frac{y^2}{2} \right]_0^1$$

$$= 2 - x - \frac{1}{2}$$

$$f_X(x) = \begin{cases} \frac{3}{2} - x, & 0 < x \le 1 \\ 0 & , \ otherwise \end{cases}$$

Marginal density function of Y is $f_Y(y) = \int_0^1 (2 - x - y)\, dx$

$$= \left[2x - \frac{x^2}{2} - xy \right]_0^1$$

$$= \frac{3}{2} - y$$

$$f_Y(y) = \begin{cases} \frac{3}{2} - y, & 0 \le y \le 1 \\ 0 & , \ otherwise \end{cases}$$

Covariance of $(X,Y) = Cov(X,Y) = E(XY) - E(X)E(Y)$

$$E(X) = \int_0^1 x f_X(x)\, dx = \int_0^1 x\left(\frac{3}{2} - x\right) dx = \left[\frac{3}{2}\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{5}{12}$$

$$E(Y) = \int_0^1 y f_Y(y)\, dy = \int_0^1 y\left(\frac{3}{2} - y\right) dy = \frac{5}{12}$$

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

$$E(XY) = \int_0^1 \int_0^1 xy\, f(x,y)\, dxdy$$

$$= \int_0^1 \int_0^1 xy(2 - x - y)\, dxdy$$

$$= \int_0^1 \int_0^1 \left(2xy - x^2 y - xy^2\right) dxdy$$

$$= \int_0^1 \left[\frac{2x^2 y}{2} - \frac{x^3}{3} y - \frac{x^2}{2} y^2 \right]_0^1 dy$$

$$= \int_0^1 \left(y - \frac{1}{3} - \frac{y^2}{2} \right) dy$$

$$= \left[\frac{y^2}{2} - \frac{y}{3} - \frac{y^3}{6} \right]_0^1 = \frac{1}{6}$$

$$Cov(X,Y) = \frac{1}{6} - \frac{5}{12} \times \frac{5}{12}$$

$$= \frac{1}{6} - \frac{25}{144} = -\frac{1}{144}.$$

b). Correlation coefficient $\rho_{XY} = \dfrac{E(XY) - E(X)E(Y)}{\sigma_X \sigma_Y}$

Marginal density function of X is

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_2^4 \left(\frac{6-x-y}{8}\right) dy = \frac{6-2x}{8}$$

Marginal density function of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^2 \left(\frac{6-x-y}{8}\right) dx = \frac{10-2y}{8}$$

Then $E(X) = \displaystyle\int_0^2 x f_X(x)\,dx = \int_0^2 x\left(\frac{6-2x}{8}\right) dx$

$$= \frac{1}{8}\left[\frac{6x^2}{2} - \frac{2x^3}{3}\right]_0^2$$

$$= \frac{1}{8}\left[12 - \frac{16}{3}\right] = \frac{1}{8} \times \frac{20}{3} = \frac{5}{6}$$

$$E(Y) = \int_2^4 y\left(\frac{10-2y}{8}\right) dy = \frac{1}{8}\left[\frac{10y^2}{2} - \frac{2y^3}{3}\right]_2^4 = \frac{17}{6}$$

$$E(X^2) = \int_0^2 x^2 f_X(x)\,dx = \int_0^2 x^2\left(\frac{6-2x}{8}\right) dx = \frac{1}{8}\left[\frac{6x^3}{3} - \frac{2x^4}{4}\right]_0^2 = 1$$

$$E(Y^2) = \int_2^4 y^2\left(\frac{10-2y}{8}\right) dy = \frac{1}{8}\left[\frac{10y^3}{3} - \frac{2y^4}{4}\right]_2^4 = \frac{25}{3}$$

$$Var(X) = \sigma_X^2 = E(X^2) - [E(X)]^2 = 1 - \left(\frac{5}{6}\right)^2 = \frac{11}{36}$$

$$Var(Y) = \sigma_Y^2 = E(Y^2) - [E(Y)]^2 = \frac{25}{3} - \left(\frac{17}{6}\right)^2 = \frac{11}{36}$$

$$E(XY) = \int_2^4\int_0^2 xy\left(\frac{6-x-y}{8}\right) dx\,dy$$

$$= \frac{1}{8}\int_2^4 \left[\frac{6x^2y}{2} - \frac{x^3y}{3} - \frac{x^2y^2}{2}\right]_0^2 dy$$

$$= \frac{1}{8}\int_2^4 \left(12y - \frac{8}{3}y - 2y^2\right) dy = \frac{1}{8}\left[\frac{12y^2}{2} - \frac{8}{3}\frac{y^2}{2} - \frac{2y^3}{3}\right]_2^4$$

$$= \frac{1}{8}\left[96 - \frac{64}{3} - \frac{128}{3} - 24 + \frac{16}{3} + \frac{16}{3}\right] = \frac{1}{8}\left[\frac{56}{3}\right]$$

$$E(XY) = \frac{7}{3}$$

$$\rho_{XY} = \frac{E(XY) - E(X)E(Y)}{\sigma_X \sigma_Y} = \frac{\frac{7}{3} - \left(\frac{5}{6}\right)\left(\frac{17}{6}\right)}{\frac{\sqrt{11}}{6}\frac{\sqrt{11}}{6}}$$

$$\rho_{XY} = -\frac{1}{11}.$$

Problem 22 a). Let the random variables X and Y have pdf

$f(x, y) = \frac{1}{3}$, $(x, y) = (0,0),(1,1),(2,0)$. Compute the correlation coefficient.

b) Let X_1 and X_2 be two independent random variables with means 5 and 10 and standard devotions 2 and 3 respectively. Obtain the correlation coefficient of UV where $U = 3X_1 + 4X_2$ and $V = 3X_1 - X_2$.

Solution:

a). The probability distribution is

Y \ X	0	1	2	$P(Y)$
0	$\frac{1}{3}$	0	0	$\frac{1}{3}$
1	0	$\frac{1}{3}$	0	$\frac{1}{3}$
0	0	0	$\frac{1}{3}$	$\frac{1}{3}$
$P(X)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	

$$E(X) = \sum_i x_i p_i(x_i) = \left(0 \times \frac{1}{3}\right) + \left(1 \times \frac{1}{3}\right) + \left(2 \times \frac{1}{3}\right) = 1$$

$$E(Y) = \sum_j y_j p_j(y_j) = \left(0 \times \frac{1}{3}\right) + \left(1 \times \frac{1}{3}\right) + \left(0 \times \frac{1}{3}\right) = \frac{1}{3}$$

$$E(X^2) = \sum_i x_i^2 p(x_i) = \left(0 \times \frac{1}{3}\right) + \left(1 \times \frac{1}{3}\right) + \left(4 \times \frac{1}{3}\right) = \frac{5}{3}$$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{5}{3} - 1 = \frac{2}{3}$$

$$E\left(Y^2\right) = \sum_j y_j^2 p\left(y_j\right) = \left(0 \times \frac{1}{3}\right) + \left(1 \times \frac{1}{3}\right) + \left(0 \times \frac{1}{3}\right) = \frac{1}{3}$$

$$\therefore V\left(Y\right) = E\left(Y^2\right) - \left[E\left(Y\right)\right]^2 = \frac{1}{3} - \frac{1}{9} = \frac{2}{9}$$

Correlation coefficient $\rho_{XY} = \dfrac{E\left(XY\right) - E\left(X\right)E\left(Y\right)}{\sqrt{V\left(X\right)V\left(Y\right)}}$

$$E\left(XY\right) = \sum_i \sum_j x_i y_j p\left(x_i, y_j\right)$$

$$= 0.0.\frac{1}{3} + 0.1.0 + 1.0.0 + 1.1.\frac{1}{3} + 1.2.0 + 0.0.0 + 0.1.0 + 0.2.\frac{1}{3} = \frac{1}{3}$$

$$\rho_{XY} = \frac{\frac{1}{3} - \left(1\right)\left(\frac{1}{3}\right)}{\sqrt{\frac{2}{3} \times \frac{2}{9}}} = 0$$

Correlation coefficient $= 0$.

b). Given $E\left(X_1\right) = 5$, $E\left(X_2\right) = 10$

$V\left(X_1\right) = 4$, $V\left(X_2\right) = 9$

Since X and Y are independent $E\left(XY\right) = E\left(X\right)E\left(Y\right)$

Correlation coefficient $= \dfrac{E\left(UV\right) - E\left(U\right)E\left(V\right)}{\sqrt{Var\left(U\right)Var\left(V\right)}}$

$$E\left(U\right) = E\left(3X_1 + 4X_2\right) = 3E\left(X_1\right) + 4E\left(X_2\right)$$

$$= \left(3 \times 5\right) + \left(4 \times 10\right) = 15 + 40 = 55.$$

$$E\left(V\right) = E\left(3X_1 - X_2\right) = 3E\left(X_1\right) - E\left(X_2\right)$$

$$= \left(3 \times 5\right) - 10 = 15 - 10 = 5$$

$$E\left(UV\right) = E\left[\left(3X_1 + 4X_2\right)\left(3X_1 - X_2\right)\right]$$

$$= E\left[9X_1^2 - 3X_1X_2 + 12X_1X_2 - 4X_2^2\right]$$

$$= 9E\left(X_1^2\right) - 3E\left(X_1X_2\right) + 12E\left(X_1X_2\right) - 4E\left(X_2^2\right)$$

$$= 9E\left(X_1^2\right) + 9E\left(X_1X_2\right) - 4E\left(X_2^2\right)$$

$$= 9E\left(X_1^2\right) + 9E\left(X_1\right)E\left(X_2\right) - 4E\left(X_2^2\right)$$

$$= 9E\left(X_1^2\right) + 450 - 4E\left(X_2^2\right)$$

$$V\left(X_1\right) = E\left(X_1^2\right) - \left[E\left(X_1\right)\right]^2$$

$$E\left(X_1^2\right) = V\left(X_1\right) + \left[E\left(X_1\right)\right]^2 = 4 + 25 = 29$$

$$E\left(X_2^2\right) = V\left(X_2\right) + \left[E\left(X_2\right)\right]^2 = 9 + 100 = 109$$

$\therefore E(UV) = (9\times 29) + 450 - (4\times 109)$

$= 261 + 450 - 436 = 275$

$Cov(U,V) = E(UV) - E(U)E(V)$

$= 275 - (5 \times 55) = 0$

Since $Cov(U,V) = 0$, Correlation coefficient $= 0$.

Problem 23. a). Let the random variable X has the marginal density function $f(x) = 1, -\frac{1}{2} < x < \frac{1}{2}$ and let the conditional density of Y be $f\left(\frac{y}{x}\right) = \begin{cases} 1, & x < y < x+1, \quad -\frac{1}{2} < x < 0 \\ 1, & -x < y < 1-x, \quad 0 < x < \frac{1}{2} \end{cases}$. Prove that the variables X and Y are uncorrelated.

b). Given $f(x,y) = xe^{-x(y+1)}, \quad x \ge 0, \quad y \ge 0$. Find the regression curve of Y on X.

Solution:

a). We have $E(X) = \int_{-\frac{1}{2}}^{\frac{1}{2}} xf(x)\,dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} x\,dx = \left[\frac{x^2}{2}\right]_{-\frac{1}{2}}^{\frac{1}{2}} = 0$

$$E(XY) = \int_{-\frac{1}{2}}^{0}\int_{x}^{x+1} xy\,dx\,dy + \int_{0}^{\frac{1}{2}}\int_{-x}^{1-x} xy\,dx\,dy$$

$$= \int_{-\frac{1}{2}}^{0} x\left[\int_{x}^{x+1} y\,dy\right]dx + \int_{0}^{\frac{1}{2}} x\left[\int_{-x}^{1-x} y\,dy\right]dx$$

$$= \frac{1}{2}\int_{-\frac{1}{2}}^{0} x(2x+1)\,dx + \frac{1}{2}\int_{0}^{\frac{1}{2}} x(1-2x)\,dx$$

$$= \frac{1}{2}\left[\frac{2x^3}{3} + \frac{x^2}{2}\right]_{-\frac{1}{2}}^{0} + \frac{1}{2}\left[\frac{x^2}{2} - \frac{2x^3}{3}\right]_{0}^{\frac{1}{2}} = 0$$

Since $Cov(X,Y) = E(XY) - E(X)E(Y) = 0$, the variables X and Y are uncorrelated.

b). Regression curve of Y on X is $E\left(\frac{y}{x}\right)$

$$E\left(\frac{y}{x}\right) = \int_{-\infty}^{\infty} yf\left(\frac{y}{x}\right)dy$$

$$f(y/x) = \frac{f(x,y)}{f_X(X)}$$

Marginal density function $f_X(x) = \int_0^{\infty} f(x, y)\, dy$

$$= x\int_0^{\infty} e^{-x(y+1)} dy$$

$$= x\left[\frac{e^{-x(y+1)}}{-x}\right]_0^{\infty} = e^{-x} \quad x \ge 0$$

Conditional pdf of Y on X is $f\left(\frac{y}{x}\right) = \frac{f(x, y)}{f_X(x)} = \frac{xe^{-xy-x}}{e^{-x}} = xe^{-xy}$

The regression curve of Y on X is given by

$$E\left(\frac{y}{x}\right) = \int_0^{\infty} yxe^{-xy} dy$$

$$= x\left[y\frac{e^{-xy}}{-x} - \frac{e^{-xy}}{x^2}\right]_0^{\infty}$$

$E\left(\frac{y}{x}\right) = \frac{1}{x} \Rightarrow y = \frac{1}{x}$ and hence $xy = 1$.

Problem 24. a). Given $f(x, y) = \begin{cases} \frac{x+y}{3}, & 0 < x < 1,\ 0 < y < 2 \\ 0 & , otherwise \end{cases}$,

obtain the regression of Y on X and X on Y.

b). Distinguish between correlation and regression Analysis

Solution:

a). Regression of Y on X is $E\left(\frac{Y}{X}\right)$

$$E\left(\frac{Y}{X}\right) = \int_{-\alpha}^{\alpha} yf\left(\frac{y}{x}\right) dy$$

$$f\left(\frac{Y}{X}\right) = \frac{f(x, y)}{f_X(x)}$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$$

$$= \int_0^2 \left(\frac{x+y}{3}\right) dy = \frac{1}{3}\left[xy + \frac{y^2}{2}\right]_0^2$$

$$= \frac{2(x+1)}{3}$$

$$f\left(\frac{Y}{X}\right) = \frac{f(x, y)}{f_X(x)} = \frac{x+y}{2(x+1)}$$

Regression of Y on $X = E\left(Y/X\right) = \int_0^2 \frac{y(x+y)}{2(x+1)}\,dy$

$$= \frac{1}{2(x+1)}\left[\frac{xy^2}{2}+\frac{y^3}{3}\right]_0^2$$

$$= \frac{1}{2(x+1)}\left[2x+\frac{8}{3}\right] = \frac{3x+4}{3(x+1)}$$

$$E\left(X/Y\right) = \int_{-\infty}^{\infty} xf\left(x/y\right)dx$$

$$f\left(x/y\right) = \frac{f(x,y)}{f_Y(y)}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_0^1 \left(\frac{x+y}{3}\right)dx = \frac{1}{3}\left[\frac{x^2}{2}+xy\right]_0^1$$

$$= \frac{1}{3}\left[\frac{1}{2}+y\right]$$

$$f\left(x/y\right) = \frac{2(x+y)}{2y+1}$$

Regression of X on Y $= E\left(X/Y\right) = \int_0^1 \frac{x+y}{2y+1}\,dx$

$$= \frac{1}{2y+1}\left[\frac{x^2}{2}+xy\right]_0^1$$

$$= \frac{\frac{1}{2}+y}{2y+1} = \frac{1}{2}.$$

b).

1. Correlation means relationship between two variables and Regression is a Mathematical Measure of expressing the average relationship between the two variables.

2. Correlation need not imply cause and effect relationship between the variables. Regression analysis clearly indicates the cause and effect relationship between Variables.

3. Correlation coefficient is symmetric i.e. $r_{xy} = r_{yx}$ where regression coefficient is not symmetric

4. Correlation coefficient is the measure of the direction and degree of linear relationship between two variables. In regression using the relationship between two variables we can predict the dependent variable value for any given independent variable value.

Problem 25. a). X any Y are two random variables with variances σ^2_x and σ^2_y respectively and r is the coefficient of correlation between them. If $U = X + KY$ and $V = X + \frac{y\sigma_x}{\sigma_y}$, find the value of k so that U and V are uncorrelated.

b). Find the regression lines:

X	6	8	10	18	20	23
Y	40	36	20	14	10	2

Solution:

Given $U = X + KY$

$$E(U) = E(X) + KE(Y)$$

$$V = X + \frac{\sigma_X}{\sigma_Y}Y$$

$$E(V) = E(X) + \frac{\sigma_X}{\sigma_Y}E(Y)$$

If U and V are uncorrelated, $Cov(U,V) = 0$

$$E\left[\left(U - E(U)\right)\left(V - E(V)\right)\right] = 0$$

$$\Rightarrow E\left[\left(X + KY - E(X) - KE(Y)\right)\times\left(X + \frac{\sigma_X}{\sigma_Y}Y - E(X) - \frac{\sigma_X}{\sigma_Y}E(Y)\right)\right] = 0$$

$$\Rightarrow E\left[\left\{\left(X - E(X)\right) + K\left(Y - E(Y)\right)\right\}\times\left\{\left(X - E(X)\right) + \frac{\sigma_X}{\sigma_Y}\left(Y - E(Y)\right)\right\}\right] = 0$$

$$\Rightarrow E\left[\left(X - E(X)\right)^2 + \frac{\sigma_X}{\sigma_Y}\left(X - E(X)\right)\left(Y - E(Y)\right) + K\left(Y - E(Y)\right)\left(X - E(X)\right) + K\frac{\sigma_X}{\sigma_Y}\left(Y - E(Y)\right)^2\right] = 0$$

$$\Rightarrow V(X) + \frac{\sigma_X}{\sigma_Y}Cov(X,Y) + KCov(X,Y) + K\frac{\sigma_X}{\sigma_Y}V(Y) = 0$$

$$K\left[Cov(X,Y) + \frac{\sigma_X}{\sigma_Y}V(Y)\right] = -V(X) - \frac{\sigma_X}{\sigma_Y}Cov(x,y)$$

$$K = \frac{-V(X) - \frac{\sigma_X}{\sigma_Y}r\sigma_X\sigma_Y}{r\sigma_X\sigma_Y + \frac{\sigma_X}{\sigma_Y}V(Y)} = \frac{-\sigma_X^2 - r\sigma_X^2}{r\sigma_X\sigma_Y + \sigma_X\sigma_Y}$$

$$= \frac{-\sigma_X^2(1+r)}{\sigma_X\sigma_Y(1+r)} = -\frac{\sigma_X}{\sigma_Y}.$$

b).

X	Y	X^2	Y^2	XY
6	40	36	1600	240
8	36	64	1296	288
10	20	100	400	200
18	14	324	196	252
20	10	400	100	200
23	2	529	4	46
$\sum X = 85$	$\sum Y = 122$	$\sum X^2 = 1453$	$\sum Y^2 = 3596$	$\sum XY = 1226$

$$\overline{X} = \frac{\sum x}{n} = \frac{85}{6} = 14.17, \quad \overline{Y} = \frac{\sum y}{n} = \frac{122}{6} = 20.33$$

$$\sigma_x = \sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2} = \sqrt{\frac{1453}{6} - \left(\frac{85}{6}\right)^2} = 6.44$$

$$\sigma_y = \sqrt{\frac{\sum y^2}{n} - \left(\frac{\sum y}{n}\right)^2} = \sqrt{\frac{3596}{6} - \left(\frac{122}{6}\right)^2} = 13.63$$

$$r = \frac{\frac{\sum xy}{n} - \overline{x}\,\overline{y}}{\sigma_x \sigma_y} = \frac{\frac{1226}{6} - (14.17)(20.33)}{(6.44)(13.63)} = -0.95$$

$$b_{xy} = r\frac{\sigma_x}{\sigma_y} = -0.95 \times \frac{6.44}{13.63} = -0.45$$

$$b_{yx} = r\frac{\sigma_y}{\sigma_x} = -0.95 \times \frac{13.63}{6.44} = -2.01$$

The regression line X on Y is

$$x - \overline{x} = b_{xy}\left(y - \overline{y}\right) \Rightarrow x - 14.17 = -0.45\left(y - \overline{y}\right)$$

$$\Rightarrow x = -0.45y + 23.32$$

The regression line Y on X is

$$y - \overline{y} = b_{yx}\left(x - \overline{x}\right) \Rightarrow y - 20.33 = -2.01\left(x - 14.17\right)$$

$$\Rightarrow y = -2.01x + 48.81$$

Problem 26. a) Using the given information given below compute $\overline{x}, \overline{y}$ and r. Also compute σ_y when $\sigma_x = 2$, $2x + 3y = 8$ and $4x + y = 10$.

b) The joint pdf of X and Y is

Y \ X	-1	1
0	$\frac{1}{8}$	$\frac{3}{8}$
1	$\frac{2}{8}$	$\frac{2}{8}$

Find the correlation coefficient of X and Y.

Solution:

a). When the regression equation are Known the arithmetic means are computed by solving the equation.

$2x + 3y = 8$ ------------- (1)

$4x + y = 10$ ------------ (2)

$(1)\times 2 \Rightarrow 4x + 6y = 16$ --------- (3)

$(2) - (3) \Rightarrow -5y = -6$

$\Rightarrow y = \frac{6}{5}$

Equation $(1) \Rightarrow 2x + 3\left(\frac{6}{5}\right) = 8$

$\Rightarrow 2x = 8 - \frac{18}{5}$

$\Rightarrow x = \frac{11}{5}$

i.e. $\overline{x} = \frac{11}{5}$ & $\overline{y} = \frac{6}{5}$

To find r, Let $2x + 3y = 8$ be the regression equation of X on Y.

$2x = 8 - 3y \Rightarrow x = 4 - \frac{3}{2}y$

$\Rightarrow b_{xy}$ = Coefficient of Y in the equation of X on $Y = -\frac{3}{2}$

Let $4x + y = 10$ be the regression equation of Y on X

$\Rightarrow y = 10 - 4x$

$\Rightarrow b_{yx}$ = coefficient of X in the equation of Y on $X = -4$.

$$r = \pm\sqrt{b_{xy}b_{yx}}$$

$$= -\sqrt{\left(-\frac{3}{2}\right)(-4)} \qquad (\because b_{xy} \text{ \& } b_{yx} \text{ are negative})$$

$$= -2.45$$

Since r is not in the range of $(-1 \le r \le 1)$ the assumption is wrong.

Now let equation (1) be the equation of Y on X

$\Rightarrow y = \frac{8}{3} - \frac{2x}{3}$

$\Rightarrow b_{yx}$ = Coefficient of X in the equation of Y on X

$b_{yx} = -\frac{2}{3}$

from equation (2) be the equation of X on Y

$b_{xy} = -\frac{1}{4}$

$r = \pm\sqrt{b_{xy} b_{yx}} = \sqrt{-\frac{2}{3} \times -\frac{1}{4}} = 0.4081$

To compute σ_y from equation (4) $b_{yx} = -\frac{2}{3}$

But we know that $b_{yx} = r\frac{\sigma_y}{\sigma_x}$

$$\Rightarrow -\frac{2}{3} = 0.4081 \times \frac{\sigma_y}{2}$$

$$\Rightarrow \sigma_y = -3.26$$

b). Marginal probability mass function of X is

When $X = 0$, $P(X) = \frac{1}{8} + \frac{3}{8} = \frac{4}{8}$

$X = 1$, $P(X) = \frac{2}{8} + \frac{2}{8} = \frac{4}{8}$

Marginal probability mass function of Y is

When $Y = -1$, $P(Y) = \frac{1}{8} + \frac{2}{8} = \frac{3}{8}$

$Y = 1$, $P(Y) = \frac{3}{8} + \frac{2}{8} = \frac{5}{8}$

$E(X) = \sum_x x\,p(x) = 0 \times \frac{4}{8} + 1 \times \frac{4}{8} = \frac{4}{8}$

$E(Y) = \sum_y y\,p(y) = -1 \times \frac{3}{8} + 1 \times \frac{5}{8} = -\frac{3}{8} + \frac{5}{8} = \frac{2}{8}$

$E(X^2) = \sum_x x^2\,p(x) = 0^2 \times \frac{4}{8} + 1^2 \times \frac{4}{8} = \frac{4}{8}$

$E(Y^2) = \sum_y y^2\,p(y) = (-1)^2 \times \frac{3}{8} + 1^2 \times \frac{5}{8} = \frac{3}{8} + \frac{5}{8} = 1$

$V(X) = E(X^2) - (E(X))^2$

$= \frac{4}{8} - \left(\frac{4}{8}\right)^2 = \frac{1}{4}$

$$V(Y) = E(Y^2) - (E(Y))^2$$
$$= 1 - \left(\frac{1}{4}\right)^2 = \frac{15}{16}$$
$$E(XY) = \sum_x \sum_y xy\, p(x,y)$$
$$= 0 \times \frac{1}{8} + 0 \times \frac{3}{8} + (-1)\frac{2}{8} + 1 \times \left(\frac{2}{8}\right) = 0$$
$$Cov(X,Y) = E(XY) - E(X)E(Y) = 0 - \frac{1}{2} \times \frac{1}{4} = -\frac{1}{8}$$
$$r = \frac{Cov(X,Y)}{\sqrt{V(X)}\sqrt{V(Y)}} = \frac{-\frac{1}{8}}{\sqrt{\frac{1}{4}}\sqrt{\frac{15}{16}}} = -0.26.$$

Problem 27. a) Calculate the correlation coefficient for the following heights (in inches) of fathers X and their sons Y.

X	65	66	67	67	68	69	70	72
Y	67	68	65	68	72	72	69	71

b) If X and Y are independent exponential variates with parameters 1, find the pdf of $U = X - Y$.

Solution:

X	Y	XY	X^2	Y^2
65	67	4355	4225	4489
66	68	4488	4359	4624
67	65	4355	4489	4285
68	72	4896	4624	5184
69	72	4968	4761	5184
70	69	4830	4900	4761
72	71	5112	5184	5041
$\sum X = 544$	$\sum Y = 552$	$\sum XY = 37560$	$\sum X^2 = 37028$	$\sum Y^2 = 38132$

$$\overline{X} = \frac{\sum x}{n} = \frac{544}{8} = 68$$
$$\overline{Y} = \frac{\sum y}{n} = \frac{552}{8} = 69$$
$$\overline{X}\,\overline{Y} = 68 \times 69 = 4692$$

$$\sigma_X = \sqrt{\frac{1}{n}\sum x^2 - \overline{X}^2} = \sqrt{\frac{1}{8}(37028) - 68^2} = \sqrt{4628.5 - 4624} = 2.121$$

$$\sigma_Y = \sqrt{\frac{1}{n}\sum y^2 - \overline{y}^2} = \sqrt{\frac{1}{8}\left(38132\right) - 69^2} = \sqrt{4766.5 - 4761} = 2.345$$

$$Cov\left(X,Y\right) = \frac{1}{n}\sum XY - \overline{X}\,\overline{Y} = \frac{1}{8}\left(37650\right) - 68\times 69$$

$$= 4695 - 4692 = 3$$

The correlation coefficient of X and Y is given by

$$r\left(X,Y\right) = \frac{Cov\left(X,Y\right)}{\sigma_X\sigma_Y} = \frac{3}{\left(2.121\right)\left(2.345\right)}$$

$$= \frac{3}{4.973} = 0.6032 .$$

b). Given that X and Y are exponential variates with parameters 1

$$f_X\left(x\right) = e^{-x},\ x \ge 0,\ f_Y\left(y\right) = e^{-y},\ y \ge 0$$

Also $f_{XY}\left(x,y\right) = f_X\left(x\right) f_Y\left(y\right)$ since X and Y are independent

$$= e^{-x}e^{-y}$$

$$= e^{-\left(x+y\right)};\quad x \ge 0,\ y \ge 0$$

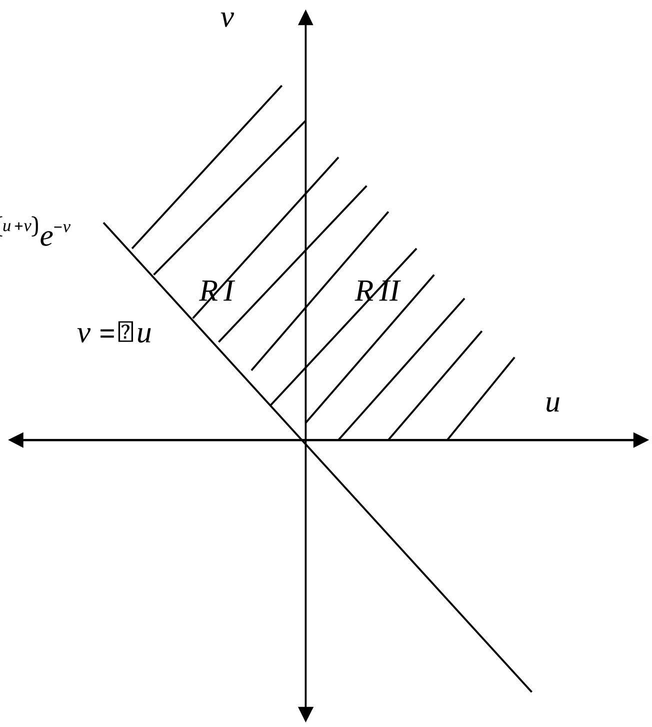
Consider the transformations $u = x - y$ and $v = y$

$\Rightarrow x = u + v,\ y = v$

$$J = \frac{\partial\left(x,y\right)}{\partial\left(u,v\right)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

$$f_{UV}\left(u,v\right) = f_{XY}\left(x,y\right)\left|J\right| = e^{-x}e^{-y} = e^{-\left(u+v\right)}e^{-v}$$

$$= e^{-\left(u+2v\right)},\ u + v \ge 0,\quad v \ge 0$$



In Region I when $u < 0$

$$f\left(u\right) = \int_{-u}^{\infty} f\left(u,v\right) dv = \int_{-u}^{\infty} e^{-u}.e^{-2v}dv$$

$$= e^{-u}\left[\frac{e^{-2v}}{-2}\right]_{-u}^{\infty}$$

$$= \frac{e^{-u}}{-2}\left[0 - e^{2u}\right] = \frac{e^{u}}{2}$$

In Region II when $u > 0$

$$f\left(u\right) = \int_{0}^{\infty} f(u,v)dv$$

$$= \int_{0}^{\infty} e^{-\left(u+2v\right)}dv = \frac{e^{-u}}{2}$$

$$\therefore f(u) = \begin{cases} \dfrac{e^{u}}{2}, & u < 0 \\ \dfrac{e^{-u}}{2}, & u > 0 \end{cases}$$

Problem 28. a) The joint pdf of X and Y is given by $f(x, y) = e^{-(x+y)}, x > 0, y > 0$. Find the pdf of $U = \dfrac{X+Y}{2}$.

b) If X and Y are independent random variables each following $N(0,2)$, find the pdf of $Z = 2X + 3Y$. If X and Y are independent rectangular variates on $(0,1)$ find the distribution of $\dfrac{X}{Y}$.

Solution:

a). Consider the transformation $u = \dfrac{x+y}{2}$ & $v = y$

$$\Rightarrow x = 2u - v \text{ and } y = v$$

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} = 2$$

$$f_{UV}(u,v) = f_{XY}(x, y)|J|$$

$$= e^{-(x+y)}2 = 2e^{-(x+y)} = 2e^{-(2u-v+v)}$$

$$= 2e^{-2u}, \ 2u - v \geq 0, \ v \geq 0$$

$$f_{UV}(u,v) = 2e^{-2u}, \ u \geq 0, \ 0 \leq v \leq \frac{u}{2}$$

$$f(u) = \int_0^{\frac{u}{2}} f_{UV}(u,v)\, dv = \int_0^{\frac{u}{2}} 2e^{-2u} dv$$

$$= \left[2e^{-2u}v\right]_0^{\frac{u}{2}}$$

$$f(u) = \begin{cases} 2\dfrac{u}{2}e^{-2u}, & u \geq 0 \\ 0, & otherwise \end{cases}$$

b).(i) Consider the transformations $w = y$,

i.e. $z = 2x + 3y$ and $w = y$

i.e. $x = \dfrac{1}{2}(z - 3w), y = w$

$$|J| = \frac{\partial(x,y)}{\partial(z,w)} = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & -\frac{3}{2} \\ 0 & 1 \end{vmatrix} = \frac{1}{2}.$$

Given that X and Y are independent random variables following $N(0,2)$

$$\therefore f_{XY}(x,y) = \frac{1}{8\pi} e^{\frac{-(x^2+y^2)}{8}}, -\infty < x, y < \infty$$

The joint pdf of (z,w) is given by

$$f_{ZW}(z,w) = |J| f_{XY}(x,y)$$

$$= \frac{1}{2} \cdot \frac{1}{8\pi} e^{-\frac{\left[\frac{1}{4}(z-3w)^2 + w^2\right]}{8}}$$

$$= \frac{1}{16\pi} e^{-\frac{1}{32}\left[(z-3w)^2 + 4w^2\right]}, -\infty < z, w < \infty.$$

The pdf of z is the marginal pdf obtained by interchanging $f_{ZW}(z,w)$ w.r.to w over the range of w.

$$\therefore f_Z(z) = \frac{1}{16\pi} \int_{-\infty}^{\infty} \left(e^{-\frac{1}{32}(z^2 - 6wz + 13w^2)} \right) dw$$

$$= \frac{1}{16\pi} e^{-\frac{z^2}{32}} \int_{-\infty}^{\infty} \left(e^{-\frac{13}{32}\left(w^2 - \frac{6wz}{13} + \left(\frac{3z}{13}\right)^2 - \left(\frac{3z}{13}\right)^2\right)} \right) dw$$

$$= \frac{1}{16\pi} e^{-\frac{z^2}{32} + \frac{9z^2}{13\times 32}} \int_{-\infty}^{\infty} \left(e^{-\frac{13}{32}\left(w - \frac{3z}{13}\right)^2} \right) dw = \frac{1}{16\pi} e^{-\frac{z^2}{8\times 13}} \int_{-\infty}^{\infty} e^{-\frac{13}{32}t^2} dt$$

$$r = \frac{13}{32}t^2 \Rightarrow dr = \frac{13}{16} t\,dt \Rightarrow \frac{16}{13t} dr = dt \Rightarrow \sqrt{\frac{r32}{13}} dr = dt$$

$$\frac{16}{13}\sqrt{\frac{13}{r32}} dr = dt \Rightarrow \frac{4}{\sqrt{13} \times \sqrt{2}} r^{-\frac{1}{2}} dr = dt$$

$$= \frac{2}{16\pi} \frac{4}{\sqrt{13} \times \sqrt{2}} e^{-\frac{z^2}{8\times 13}} \int_0^{\infty} e^{-r} r^{-\frac{1}{2}} dr$$

$$= \frac{1}{2\pi \sqrt{13} \times \sqrt{2}} e^{-\frac{z^2}{8\times 13}} \int_0^{\infty} e^{-r} r^{-\frac{1}{2}} dr = \frac{1}{2\pi \sqrt{13} \times \sqrt{2}} e^{-\frac{z^2}{8\times 13}} \sqrt{\pi} = \frac{1}{2\sqrt{13}\sqrt{2\pi}} e^{-\frac{z^2}{2\left(2\sqrt{13}\right)^2}}$$

i.e. $Z \sim N\left(0, 2\sqrt{13}\right)$

b).(ii) Given that X and Y are uniform Variants over $(0,1)$

$$\therefore f_X(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & otherwise \end{cases} \text{ and } f_Y(y) = \begin{cases} 1, & 0 < y < 1 \\ 0, & otherwise \end{cases}$$

Since X and Y are independent,

$$f_{XY}(x,y) = f_X(x) f_y(y) \begin{cases} 1, & 0 < x, y < 1 \\ 0, & otherwise \end{cases}$$

Consider the transformation $u = \frac{x}{y}$ and $v = y$

i.e. $x = uv$ and $y = v$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & 0 \\ u & 1 \end{vmatrix} = v$$

$$\therefore f_{UV}(u,v) = f_{XY}(x,y)|J|$$

$$= v,\ 0 < u < \infty,\ 0 < v < \infty$$

The range for u and v are identified as follows.

$0 < x < 1$ and $0 < y < 1$.

$\Rightarrow 0 < uv < 1$ and $0 < v < 1$

$\Rightarrow uv > 0,\ uv < 1,\ v > 0$ and $v < 1$

$\Rightarrow uv > 0$ and $v > 0 \Rightarrow u > 0$

Now $f(u) = \int f_{UV}(u,v)\,dv$

The range for v differs in two regions

$$f(u) = \int_0^1 f_{UV}(u,v)dv$$

$$= \int_0^1 v\,dv = \left[\frac{v^2}{2}\right]_0^1 = \frac{1}{2},\ 0 < u < 1$$

$$f(u) = \int_0^{\frac{1}{u}} f_{UV}(u,v)dv = \int_0^{\frac{1}{u}} v\,dv = \left[\frac{v^2}{2}\right]_0^{\frac{1}{u}} = \frac{1}{2u^2},\ 1 \le u \le \infty$$

$$\therefore f(u) = \begin{cases} \frac{1}{2}, & 0 \le u \le 1 \\ \frac{1}{2u^2}, & u > 1 \end{cases}$$

Problem 29. a) If $X_1, X_2, ...X_n$ are Poisson variates with parameter $\lambda = 2$. Use the central limit theorem to estimate $P(120 < S_n < 160)$ where $s_n = X_1 + X_2 + ... + X_n$ and $n = 75$.

b) A random sample of size 100 is taken from a population whose mean is 60 and variance is 400. Using central limit theorem, with what probability can we assent that the mean of the sample will not differ from $\mu = 60$ by more than 4.

Solution:

a). Given that $E(X_i) = \lambda = 2$ and $Var(X_i) = \lambda = 2$

[Since in Poisson distribution mean and variance are equal to λ]
i.e. $\mu = 2$ and $\sigma^2 = 2$

By central limit theorem, $S_n \sim N\left(n\mu, n\sigma^2\right)$

$$S_n \sim N(150,150)$$

$$\therefore P(120 < S_n < 160) = P\left(\frac{120-150}{\sqrt{150}} < z < \frac{160-150}{\sqrt{150}}\right)$$

$$= P(-2.45 < z < 0.85)$$

$$= P(-2.45 < z < 0) + P(0 < z < 0.85)$$

$$= P(0 < z < 2.45) + P(0 < z < 0.85) \quad = 0.4927 + 0.2939 = 0.7866$$

b). Given that $n = 100$, $\mu = 60$, $\sigma^2 = 400$

Since the probability statement is with respect to mean, we use the Linderberg-levy form of central limit Theorem.

$\overline{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ i.e. $\overline{X}$ follows normal distribution with mean 'μ' and variance $\frac{\sigma^2}{n}$.

i.e. $\overline{X} \sim N\left(60, \frac{400}{100}\right)$

$$\overline{X} \sim N(60,4)$$

$$P\left[\begin{matrix}\text{mean of the sample will not} \\ \text{differ from 60 by more than 4}\end{matrix}\right] = P\left[\begin{matrix}\overline{X} \text{ will not differ from} \\ \mu = 60 \text{ by more than 4}\end{matrix}\right]$$

$$= P\left(\left|\overline{X} - \mu\right| \le 4\right)$$

$$= P\left[-4 \le \overline{X} - \mu \le 4\right]$$

$$= P\left[-4 \le \overline{X} - 60 \le 4\right]$$

$$= P\left[56 \le \overline{X} \le 64\right] = P\left[\frac{56-60}{2} \le z \le \frac{64-60}{2}\right]$$

$$= P[-2 \le Z \le 2]$$

$$= 2P[0 \le Z \le 2] = 2 \times 0.4773 = 0.9446$$

Problem 30 a) If the variable X_1, X_2, X_3, X_4 are independent uniform variates in the interval $(450, 550)$, find $P(1900 \le X_1 + X_2 + X_3 + X_4 \le 2100)$ using central limit theorem.

b) A distribution with unknown mean μ has a variance equal to 1.5. Use central limit theorem to find how large a sample should be taken from the distribution in order that the probability will be at least 0.95 that the sample mean will be within 0.5 of the population mean.

Solution

a). Given that X follows a uniform distribution in the interval $(450, 550)$

$\text{Mean} = \dfrac{b+a}{2} = \dfrac{450+550}{2} = 500$

$\text{Variance} = \dfrac{(b-a)^2}{12} = \dfrac{(550-450)^2}{12} = 833.33$

By CLT $S_n = X_1 + X_2 + X_3 + X_4$ follows a normal distribution with $N\left(n\mu, n\sigma^2\right)$

The standard normal variable is given by $Z = \dfrac{S_n - n\mu}{n\sigma^2}$

when $S_n = 1900$, $Z = \dfrac{1900 - 4\times 500}{\sqrt{4\times 833.33}} = -\dfrac{100}{57.73} = -1.732$

when $S_n = 2100$, $Z = \dfrac{2100-2000}{\sqrt{4\times 833.33}} = \dfrac{100}{57.73} = 1.732$

$\therefore P\left(1900 \le S_n \le 2100\right) = P\left(-1.732 < z < 1.732\right)$

$= 2\times P\left(0 < z < 1.732\right) = 2\times 0.4582 = 0.9164$.

b). Given $E\left(X_i\right) = \mu$ and $Var\left(X_i\right) = 1.5$

Let $\overline{X}$ denote the sample mean.

By C.L.T. $\overline{X}$ follows $N\left(\mu, \dfrac{\sqrt{1.5}}{\sqrt{n}}\right)$

We have to find 'n' such that $P\left(\mu - 0.5 < \overline{X} < \mu + 0.5\right) \ge 0.95$

i.e. $P\left(-0.5 < \overline{X} - \mu < 0.5\right) \ge .95$

$P\left(\left|\overline{X} - \mu\right| < 0.5\right) \ge .95$

$P\left[\left|\dfrac{z}{\frac{\sigma}{\sqrt{n}}}\right| < 0.5\right] \ge 0.95$

$P\left[\left|z\right| < 0.5\dfrac{\sqrt{n}}{\sigma}\right] \ge 0.95$

$P\left[\left|z\right| < 0.5\dfrac{\sqrt{n}}{\sqrt{1.5}}\right] \ge 0.95$

ie $P\left(\left|Z\right| < 0.4082\sqrt{n}\right) \ge 0.95$

Where 'Z' is the standard normal variable.

The Last value of 'n' is obtained from $P\left(\left|Z\right| < 0.4082\sqrt{n}\right) = 0.95$

$2P\left(0 < z < 0.4082\sqrt{n}\right) = 0.95$

$\Rightarrow 0.4082\sqrt{n} = 1.96 \Rightarrow n = 23.05$

$\therefore$The size of the sample must be atleast 24.

UNIT – IV CORRELATION AND SPECTRAL DENSITIES

PART-A

Problem1. Define autocorrelation function and prove that for a WSS process $\{X(t)\}$, $R_{XX}(-\tau) = R_{XX}(\tau)$ 2.State any two properties of an autocorrelation function.

Solution:

Let $\{X(t)\}$ be a random process. Then the auto correlation function of the process $\{X(t)\}$ is the expected value of the product of any two members $X(t_1)$ and $X(t_2)$ of the process and is given by $R_{XX}(t_1,t_2) = E\left[X(t_1)X(t_2)\right]$ or $R_{XX}(t,t+\tau) = E\left[X(t)X(t+\tau)\right]$

For a WSS process $\{X(t)\}$, $R_{XX}(\tau) = E\left[X(t)X(t-\tau)\right]$

$\therefore R_{XX}(-\tau) = E\left[X(t)X(t+\tau)\right] = E\left[X(t+\tau)X(t)\right] = R(t+\tau-t) = R(\tau)$

Problem 2. State any two properties of an autocorrelation function.

Solution:

The process $\{X(t)\}$ is stationary with autocorrelation function $R(\tau)$ then

(i) $R(\tau)$ is an even function of τ

(ii) $R(\tau)$ is maximum at $\tau = 0$ i.e., $|R(\tau)| \le R(0)$

Problem 3. Given that the autocorrelation function for a stationary ergodic process with no periodic components is $R(\tau) = 25 + \frac{4}{1+6\tau^2}$. Find the mean and variance of the process $\{X(t)\}$.

Solution:

$$\mu_x^2 = \underset{\tau\to\infty}{Lt} R(\tau) = \underset{\tau\to\infty}{Lt} 25 + \frac{4}{1+6\tau^2} = 25$$

$\therefore \mu_x = 5$

$E\left(X^2(t)\right) = R_{XX}(0) = 25 + 4 = 29$

$Var\left(X(t)\right) = E\left[X^2(t)\right] - E\left[X(t)\right]^2 = 29 - 25 = 4$

Problem 4. Find the mean and variance of the stationary process $\{X(t)\}$ whose autocorrelation function $R(\tau) = \frac{25\tau^2+36}{6.25\tau^2+4}$.

Solution:

$$R(\tau) = \frac{25\tau^2 + 36}{6.25\tau^2 + 4} = \frac{25 + \frac{36}{\tau^2}}{6.25 + \frac{4}{\tau^2}}$$

$$\mu_x^2 = \underset{\tau \to \infty}{Lt} R(\tau) = \frac{25}{6.25} = \frac{2500}{625} = 4$$

$$\mu_x = 2$$

$$E\left[X^2(t)\right] = R_{XX}(0) = \frac{36}{4} = 9$$

$$Var\left[X(t)\right] = \{E[X(t)]\}^2 - E[X(t)]^2 = 9 - 4 = 5$$

Problem 5. Define cross-correlation function and mention two properties.
Solution:

The cross-correlation of the two process $\{X(t)\}$ and $\{Y(t)\}$ is defined by $R_{XY}(t_1, t_2) = E\left[X(t_1)Y(t_2)\right]$

Properties: The process $\{X(t)\}$ and $\{Y(t)\}$ are jointly wide-sense stationary with the cross-correlation function $R_{XY}(\tau)$ then

(i) $R_{YX}(\tau) = R_{XY}(-\tau)$

(ii) $|R_{XY}(\tau)| \le \sqrt{R_{XX}(0) R_{YY}(0)}$

Problem 6. Find the mean – square value of the process $\{X(t)\}$ if its autocorrelation function is given by $R(\tau) = e^{-\tau^2/2}$.
Solution:

$$\text{Mean-Square value} = E\left[X^2(t)\right] = R_{XX}(0) = \left(e^{-\frac{\tau^2}{2}}\right)_{\tau=0} = 1$$

Problem 7. Define the power spectral density function (or spectral density or power spectrum) of a stationary process?
Solution:

If $\{X(t)\}$ is a stationary process (either in the strict sense or wide sense with auto correlation function $R(\tau)$, then the Fourier transform of $R(\tau)$ is called the power spectral density function of $\{X(t)\}$ and denoted by $S_{XX}(\omega)$ or $S(\omega)$ or $S_X(\omega)$

$$\text{Thus } S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau$$

Problem 8. State any two properties of the power spectral density function.
Solution:
(i). The spectral density function of a real random process is an even function.

(ii). The spectral density of a process $\{X(t)\}$, real or complex, is a real function of ω and non-negative.

Problem 9. State Wiener-Khinchine Theorem.

Solution:

If $X_T(\omega)$ is the Fourier transform of the truncated random process defined as

$$X_T(t) = \begin{cases} X(t) & for\ |t| \le T \\ 0 & for\ |t| > T \end{cases}$$

Where $\{X(t)\}$ is a real WSS Process with power spectral density function $S(\omega)$, then

$$S(\omega) = \underset{T\to\infty}{Lt}\left[\frac{1}{2T}E\left\{\left|X_T(\omega)\right|^2\right\}\right]$$

Problem 10. If $R(\tau) = e^{-2\lambda|\tau|}$ is the auto Correlation function of a random process $\{X(t)\}$, obtain the spectral density of $\{X(t)\}$.

Solution:

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-i\omega\tau}d\tau$$

$$= \int_{-\infty}^{\infty} e^{-2\lambda|\tau|}\left(\cos\omega\tau - i\sin\omega\tau\right)d\tau$$

$$= 2\int_{0}^{\infty} e^{-2\lambda\tau}\cos\omega\tau\, d\tau$$

$$= \left[\frac{2e^{-2\lambda\tau}}{4\lambda^2+\omega^2}\left(-2\lambda\cos\omega\tau + \omega\sin\omega\tau\right)\right]_0^{\infty}$$

$$S(\omega) = \frac{4\lambda}{4\lambda^2+\omega^2}$$

Problem 11. The Power spectral density of a random process $\{X(t)\}$ is given by

$$S_{XX}(\omega) = \begin{cases} \pi, & |\omega| < 1 \\ 0, & elsewhere \end{cases}$$ Find its autocorrelation function.

Solution:

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{i\omega\tau}d\omega$$

$$= \frac{1}{2\pi}\int_{-1}^{1} \pi e^{i\omega\tau}d\omega$$

$$= \frac{1}{2}\left[\frac{e^{i\omega\tau}}{i\tau}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\frac{e^{i\tau} - e^{-i\tau}}{i\tau}\right]$$

$$= \frac{1}{\tau}\left[\frac{e^{i\tau} - e^{-i\tau}}{2i}\right] = \frac{\sin\tau}{\tau}$$

Problem 12. Define cross-Spectral density.
Solution:

The process $\{X(t)\}$ and $\{Y(t)\}$ are jointly wide-sense stationary with the cross-correlation function $R_{XY}(\tau)$, then the Fourier transform of $R_{XY}(T)$ is called the cross spectral density function of $\{X(t)\}$ and $\{Y(t)\}$ denoted as $S_{XY}(\omega)$

Thus $S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-i\omega\tau} d\tau$

Problem 13. Find the auto correlation function of a stationary process whose power spectral density function is given by $s(\omega) = \begin{cases} \omega^2 & for \ |\omega| \le 1 \\ 0 & for \ |\omega| > 1 \end{cases}$.

Solution:

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega) e^{i\omega\tau} d\omega$$

$$= \frac{1}{2\pi}\int_{-1}^{1} \omega^2 (\cos\omega\tau + i\sin\omega\tau) d\omega$$

$$= \int_0^1 \omega^2 . \cos\omega\tau \, d\omega = \frac{1}{\pi}\left[\omega^2\left(\frac{\sin\omega\tau}{\tau}\right) - 2\omega\left(\frac{-\cos\omega\tau}{\tau^2}\right) + 2\left(\frac{-\sin\omega\tau}{\tau^3}\right)\right]_0^1$$

$$R(\tau) = \frac{1}{\pi}\left[\frac{-\sin\tau}{\tau} + \frac{2\cos\tau}{\tau^2} - \frac{2\sin\tau}{\tau^3}\right]$$

Problem 14. Given the power spectral density : $S_{xx}(\omega) = \frac{1}{4+\omega^2}$, find the average power of the process.

Solution:

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega) e^{i\omega\tau} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{4+\omega^2}\right] e^{i\omega\tau} d\omega$$

Hence the average power of the processes is given by

$$E\left[X^2(t)\right] = R(0)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega}{4+\omega^2}$$

$$= \frac{1}{2\pi} 2\int_0^{\infty}\frac{d\omega}{2^2+\omega^2}$$

$$= \frac{1}{\pi}\left[\frac{1}{2}\tan^{-1}\left(\frac{\omega}{2}\right)\right]_0^{\infty}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{4} - 0\right] = \frac{1}{4}.$$

Problem 15. Find the power spectral density of a random signal with autocorrelation function $e^{-\lambda|\tau|}$.

Solution:

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} e^{-\lambda|\tau|}(\cos\omega\tau - i\sin\omega\tau)\, d\tau$$

$$= 2\int_{0}^{\infty} e^{-\lambda|\tau|}\cos\omega\tau\, d\tau$$

$$= 2\left[\frac{e^{-\lambda\tau}}{\lambda^2+\omega^2}(-\lambda\cos\omega\tau + \omega\sin\omega\tau)\right]_0^{\infty}$$

$$= 2\left[0 - \frac{1}{\lambda^2+\omega^2}(-\lambda)\right] = \frac{2\lambda}{\lambda^2+\omega^2}$$

PART-B

Problem 16. a). If $\{X(t)\}$ is a W.S.S. process with autocorrelation function $R_{XX}(\tau)$ and if $Y(t) = X(t+a) - X(t-a)$ .Show that $R_{YY}(\tau) = 2R_{XX}(\tau) - R_{XX}(\tau+2a) - R_{XX}(\tau-2a)$.

Solution:

$$R_{yy}(\tau) = E[y(t)y(t+\tau)]$$

$$= E\{[X(t+a) - X(t-a)][X(t+\tau+a) - X(t+\tau-a)]\}$$

$$= E[X(t+a)X(t+\tau+a)] - E[X(t+a)X(t+\tau-a)]$$

$$- E[X(t-a)X(t+\tau+a)] + E[X(t-a)X(t+\tau-a)]$$

$$= R_{xx}(\tau) - E[X(t+a)X(t+a+\tau-2a)]$$

$$- E[X(t-a)X(t-a+\tau+2a)] + R_{xx}(\tau)$$

$$= 2R_{xx}(\tau) - R_{xx}(\tau-2a) - R_{xx}(\tau+2a)$$

b). Assume a random signal $Y(t) = X(t) + X(t-a)$ where $X(t)$ is a random signal and $'a'$ is a constant. Find $R_{YY}(\tau)$.

Solution:

$$R_{YY}(\tau) = E[Y(t)Y(t+\tau)]$$

$$= E\{[X(t) + X(t-a)][X(t+\tau) + X(t+\tau-a)]\}$$

$$= E\left[X(t)X(t+\tau)\right] + E\left[X(t)X(t+\tau-a)\right]$$
$$+ E\left[X(t-a)X(t+\tau)\right] + E\left[X(t-a)X(t+\tau-a)\right]$$
$$= R_{xx}(\tau) + R_{xx}(\tau+a) + R_{xx}(\tau-a) + R_{xx}(\tau)$$
$$R_{yy}(\tau) = 2R_{xx}(\tau) + R_{xx}(\tau+a) + R_{xx}(\tau-a)$$

Problem 17. a). If $\{X(t)\}$ and $\{Y(t)\}$ are independent WSS Processes with zero means, find the autocorrelation function of $\{Z(t)\}$, when (*i*) $Z(t) = a + bX(t) + CY(t)$, (*ii*) $Z(t) = aX(t)Y(t)$.

Solution:

Given $E\left[X(t)\right] = 0$, $E\left[Y(t)\right] = 0$ -------- (1)

$\because \{X(t)\}$ and $\{Y(t)\}$ are independent

$E\{X(t)Y(t)\} = E\left[X(t)Y(t)\right] = 0$ --------(2)

(i). $R_{ZZ}(\tau) = E\left[Z(t)Z(t+\tau)\right]$

$$= E\{\left[a + bX(t) + cY(t)\right]\left[a + bX(t+\tau) + cY(t+\tau)\right]\}$$
$$= E\{a^2 + ab\,X(t+\tau) + acY(t+\tau) + baX(t) + b^2X(t)X(t+\tau)$$
$$+ bc\,X(t)Y(t+\tau) + caY(t) + cbY(t)X(t+\tau) + c^2y(t)y(t+\tau)\}$$
$$= E(a^2) + abE\left[X(t+\tau)\right] + acE\left[Y(t+\tau)\right] + baE\left[X(t)\right] + b\,E\left[X(t)X(t+\tau)\right]$$
$$+ bc\,E\left[X(t)y(t+\tau)\right] + caE\left[Y(t)\right] + cbE\left[Y(t)X(t+\tau)\right] + c^2E\left[Y(t)Y(t+\tau)\right]$$
$$= a^2 + b^2R_{XX}(\tau) + c^2R_{YY}(\tau)$$

$R_{ZZ}(\tau) = E\left[Z(t)Z(t+\tau)\right]$

$$= E\left[aX(t)Y(t)aX(t+\tau)Y(t+\tau)\right]$$
$$= E\left[a^2X(t)X(t+\tau)Y(t)Y(t+\tau)\right]$$
$$= a^2E\left[X(t)X(t+\tau)\right]E\left[Y(t)Y(t+\tau)\right]$$
$$R_{ZZ}(\tau) = a^2R_{XX}(\tau)R_{YY}(\tau)$$

b). If $\{X(t)\}$ is a random process with mean 3 and autocorrelation $R_{xx}(\tau) = 9 + 4e^{-0.2|\tau|}$. Determine the mean, variance and covariance of the random variables $Y = X(5)$ and $Z = X(8)$.

Solution:

Given $Y = X(5)$ & $Z = X(8)$, $E\left[X(t)\right] = 3$------------(1)

Mean of $Y = E[Y] = E\left[X(5)\right] = 3$

Mean of $Z = E[Z] = E\left[X(8)\right] = 3$

We know that

$Var(Y) = E(Y^2) - [E(Y)]^2$

$E(Y^2) = E(X^2(5))$

But $E[X^2(t)] = R_{XX}(0)$

$= 9 + 4e^{-0.2|0|}$

$= 9 + 4 = 13$

Thus $Var(Y) = 13 - (3)^2 = 13 - 9 = 4$

$Var(Z) = E(Z^2) - [E(Z)]^2$

$E[Z^2] = E[X^2(8)] \quad [\because Z = X(8)]$

$= R_{XX}(0)$

$= 9 + 4 = 13$

Hence $Var(Z) = 13 - (3^2) = 13 - 9 = 4$

$E[YZ] = R(5,8) = 9 + 4e^{-0.2|5-8|} \quad \left(\because R(t_1, t_2) = 9 + 4e^{-0.2|t_1 - t_2|}\right)$

$= 9 + 4e^{-0.6}$

Covariance $= R(t_1, t_2) - E[X(t_1)]\, E[X(t_2)]$

$= R(5,8) - E[5]E[8]$

$= 9 + 4e^{-0.6} - (3 \times 3) = 4e^{-0.6} = 2.195$

Problem 18. a). The autocorrelation function for a stationary process is given by $R_{xx}(\tau) = 9 + 2e^{-|\tau|}$. Find the mean value of the random variable $Y = \int_0^2 X(t)\,dt$ and variance of $X(t)$.

Solution:

Given $R_{xx}(\tau) = 9 + 2e^{|\tau|}$

Mean of $X(t)$ is given by

$\overline{X}^2 = E[X(t)]^2 = \underset{|\tau| \to \infty}{Lt} R_{xx}(\tau)$

$= \underset{|\tau| \to \infty}{Lt} \left(9 + 2e^{-|\tau|}\right)$

$\overline{X}^2 = 9$

$\overline{X} = 3$

Also $E[X^2(t)] = R_{XX}(0) = 9 + 2e^{-|0|} = 9 + 2 = 11$

$Var\{X(t)\} = E[X^2(t)] - [E(X(t))]^2$

$= 11 - 3^2 = 11 - 9 = 2$

Mean of $Y(t) = E[Y(t)]$

$$= E\left[\int_0^2 X(t)\,dt\right]$$

$$= \int_0^2 E\left[X(t)\right] dt$$

$$= \int_0^2 3dt = 3(t)_0^2 = 6$$

$$\therefore E\left[Y(t)\right] = 6$$

b). Find the mean and autocorrelation function of a semi random telegraph signal process.

Solution:

Semi random telegraph signal process.

If $N(t)$ represents the number of occurrences of a specified event in $(0,t)$ and $X(t) = (-1)^{N(t)}$, then $\{X(t)\}$ is called the semi random signal process and $N(t)$ is a poisson process with rate λ.

By the above definition $X(t)$ can take the values $=1$ and -1 only

$$P\left[X(t) = 1\right] = P\left[N(t)\, is\, even\right]$$

$$= \sum_{K=even} \frac{e^{-\lambda t}(\lambda t)^k}{K!}$$

$$= e^{-\lambda t}\left[1 + \frac{(\lambda t)^2}{2} + \ldots\right]$$

$$P\left[X(t) = 1\right] = e^{-\lambda t} cosh\lambda t$$

$$P\left[X(t) = -1\right] = P\left[N(t)\, is\, odd\right]$$

$$= \sum_{K=odd} \frac{e^{-\lambda t}(\lambda t)^k}{K!}$$

$$= e^{-\lambda t}\left[\lambda t + \frac{(\lambda t)^3}{3!} + \ldots\right]$$

$$= e^{-\lambda t} sinh\lambda t$$

$$Mean\{X(t)\} = \sum_{K=-1,1} K\, P\left(X(t) = K\right)$$

$$= 1 \times e^{-\lambda t} cosh\lambda t + (-1) \times e^{-\lambda t} sinh\lambda t$$

$$= e^{-\lambda t}\left[cosh\lambda t - sinh\lambda t\right]$$

$$= e^{-2\lambda t} \quad \left[\because cosh\lambda t - sinh\lambda t = e^{-\lambda t}\right]$$

$$R(\tau) = E\left[X(t)\,X(t+\tau)\right]$$

$$= 1 \times P\left[X(t)\,X(t+\tau) = 1\right] + -1 \times P\left[X(t)\,X(t+\tau) = -1\right]$$

$$= \sum_{n=even} e^{\lambda t} \frac{(\lambda\tau)^n}{n!} - \sum_{n=odd} e^{-\lambda t} \frac{(\lambda\tau)^n}{n!}$$
$$= e^{-\lambda\tau} cosh\lambda\tau - e^{\lambda\tau} sinh\lambda\tau$$
$$= e^{-\lambda\tau}\left[cosh\lambda\tau - sinh\lambda\tau\right]$$
$$= e^{-\lambda\tau} e^{-\lambda\tau}$$
$$R(\tau) = e^{-2\lambda\tau}$$

Problem 19. a). Find Given the power spectral density of a continuous process as $S_{XX}(\omega) = \dfrac{\omega^2 + 9}{\omega^4 + 5\omega^2 + 4}$

find the mean square value of the process.

Solution:

We know that mean square value of $\{X(t)\}$

$$= E\{X^2(t)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\,d\omega$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\omega^2 + 9}{(\omega^4 + 5\omega^2 + 4)}d\omega$$
$$= \frac{1}{2\pi}.2\int_{0}^{\infty} \frac{\omega^2 + 9}{\omega^4 + 5\omega^2 + 4}d\omega$$
$$= \frac{1}{\pi}\int_{0}^{\infty} \frac{\omega^2 + 9}{\omega^4 + \omega^2 + 4\omega^2 + 4}d\omega$$
$$= \frac{1}{\pi}\int_{0}^{\infty} \frac{\omega^2 + 9}{\omega^2(\omega^2 + 1) + 4(\omega^2 + 1)}d\omega$$

i.e., $E\{X^2(t)\} = \dfrac{1}{\pi}\displaystyle\int_{0}^{\infty} \frac{\omega^2 + 9}{(\omega^2 + 4)(\omega^2 + 1)}d\omega$

let $\omega^2 = u$

$\therefore$ We have $\dfrac{\omega^2 + 9}{(\omega^2 + 4)(\omega^2 + 1)} = \dfrac{u + 9}{(u + 4)(u + 1)}$

$$= \frac{\frac{-4 + 9}{-4 + 1}}{u + 4} + \frac{\frac{-1 + 9}{-1 + 4}}{u + 1} = -\frac{5}{3(u + 4)} + \frac{8}{3(u + 1)}$$ ….Partial fractions

i.e., $\dfrac{\omega^2 + 9}{(\omega^2 + 4)(\omega^2 + 1)} = -\dfrac{5}{3(\omega^2 + 4)} + \dfrac{8}{3(\omega^2 + 1)}$

$\therefore$From (1),

$$E\{X^2(t)\} = \frac{1}{\pi}\int_{0}^{\infty} \frac{1}{3}\left[\frac{-5}{(\omega^2 + 4)} + \frac{8}{(\omega^2 + 1)}\right]d\omega$$

$$= \frac{1}{3\pi}\left[-5.\frac{1}{2}\tan^{-1}\frac{\omega}{2} + 8\tan^{-1}\omega\right]_0^{\infty}$$

$$= \frac{1}{3\pi}\left[-\frac{5}{2}\left(\frac{\pi}{2}\right) + 8\left(\frac{\pi}{2}\right) - 0\right]$$

$$= \frac{1}{3\pi}\cdot\frac{\pi}{2}\left(-\frac{5}{2} + 8\right) = \frac{1}{6}\left(\frac{11}{2}\right)$$

$$E\{X^2(t)\} = \frac{11}{12}.$$

b). If the $2n$ random variables A_r and B_r are uncorrelated with zero mean and $E(A_r^2) = E(B_r^2) = \sigma_r^2$. Find the mean and autocorrelation of the process $X(t) = \sum_{r=1}^{n} A_r \cos\omega_r t + B_r \sin\omega_r t$.

Solution:

Given $E(A_r) = E(B_r) = 0$ & $E(A_r^2) = E(B_r^2) = \sigma_r^2$

Mean: $E[X(t)] = \left[\sum_{r=1}^{n} A_r \cos\omega_r t + B_r \sin\omega_r t\right]$

$$= \sum_{r=1}^{n}\left[E(A_r)\cos\omega_r t + E(B_r)\sin\omega_r t\right]$$

$E[X(t)] = 0 \qquad \because E(A_r) = E(B_r) = 0$

Autocorrelation function:

$$R(\tau) = E[X(t)X(t+\tau)]$$

$$= E\left\{\sum_{r=1}^{n}\sum_{s=1}^{n}(A_r\cos\omega_r t + B_r\sin\omega_r t)(A_s\cos\omega_s(t+\tau) + B_s\sin\omega_s(t+\tau))\right\}$$

Given $2n$ random variables A_r and B_r are uncorrelated

$E[A_r A_s], E[A_r B_s], E[B_r A_s], E[B_r, B_s]$ are all zero for $r \neq s$

$$= \sum_{r=1}^{n} E(A_r^2)\cos\omega_r t\cos\omega_r(t+\tau) + E(B_r^2)\sin\omega_r t\sin\omega_r(t+\tau)$$

$$= \sum_{r=1}^{n}\sigma_r^2\cos\omega_r(t-t-\tau)$$

$$= \sum_{r=1}^{n}\sigma_r^2\cos\omega_r(-\tau)$$

$$R(\tau) = \sum_{r=1}^{n}\sigma_r^2\cos\omega_r\tau$$

Problem 20. a). If $\{X(t)\}$ is a WSS process with autocorrelation $R(\tau) = Ae^{-\alpha|\tau|}$, determine the second – order moment of the random variable $X(8) - X(5)$.

Solution:

Given $R(\tau) = Ae^{-\alpha|\tau|}$

$R(t_1, t_2) = Ae^{-\alpha|t_1 - t_2|}$

$E\left[X^2(t)\right] = R_{XX}(0) = Ae^{-\alpha|0|} = A$

$\therefore E\left[X(8) - X(5)\right]^2 = E\left[X^2(8)\right] + E\left[X^2(5) - 2E\left[X(8)X(5)\right]\right]$

$E\left[X^2(8)\right] = E\left[X^2(5)\right] = A$

Also $E\left[X(8)X(5)\right] = R(8,5) = Ae^{-\alpha|8-5|} = Ae^{-3\alpha}$

Substituting (2), (3) in (1) we obtain

$E\left[X(8) - X(5)\right]^2 = A + A - 2Ae^{-3\alpha}$

$$= 2A - 2Ae^{-3\alpha}$$
$$= 2A\left(1 - e^{-3\alpha}\right)$$

b). Two random process $\{X(t)\}$ and $\{Y(t)\}$ are given by $X(t) = A\cos(\omega t + \theta), Y(t) = A\sin(\omega t + \theta)$ where A & ω are constants and θ is a uniform random variable over 0 to 2π. Find the cross-correlation function.

Solution:

$R_{XY}(\tau) = E\left[X(t)Y(t+\tau)\right]$

$$= E\left[A\cos(\omega t + \theta)\,A\sin(\omega(t+\tau) + \theta)\right]$$
$$= A^2 E\left[\sin(\omega t + \omega\tau + \theta)\cos(\omega t + \theta)\right]$$

$\because \theta$ is a uniform random variables $\quad f_\theta(\theta) = \frac{1}{2\pi}, \; 0 \le \theta \le 2\pi$

$\therefore R_{XY}(\tau) = \frac{A^2}{2\pi}\int_0^{2\pi} \sin(\omega t + \omega\tau + \theta)\cos(\omega t + \theta)\, d\theta$

$$= \frac{A^2}{2\pi}\int_0^{2\pi}\left(\frac{\sin(2\omega t + \omega\tau + 2\theta) + \sin(\omega t)}{2}\right)d\theta$$

$$= \frac{A^2}{4\pi}\left[-\frac{\cos(2\omega t + \omega\tau + 2\theta)}{2} + \theta\sin(\omega t)\right]_0^{2\pi}$$

$$= \frac{A^2}{4\pi}\left[0 + 2\pi\sin\omega\tau\right]$$

$$= \frac{A^2}{2}\sin\omega\tau$$

Problem 21. a). Find the cross-correlation function of $W(t) = A(t) + B(t)$ & $Z(t) = A(t) - B(t)$ where $A(t)$ and $B(t)$ are statistically independent random variables with zero means and autocorrelation function $R_{AA}(\tau) = e^{-|\tau|}, -\infty < \tau < \infty$, $R_{BB}(\tau) = 3e^{-|\tau|}\; -\infty < \tau < \infty$ respectively.

Solution:

Given $E\left[A(t)\right]=0, E\left[B(t)\right]=0$

$\because$ $A(t)$ & $B(t)$ are independent

$R_{AB}(\tau)=E\left[A(t)B(t+\tau)\right]=E\left[A(t)\right]E\left[B(t+\tau)\right]=0$

Similarly $R_{BA}(\tau)=E\left[B(t)A(t+\tau)\right]=E\left[B(t)\right]E\left[A(t+\tau)\right]=0$

$\therefore R_{WZ}(\tau)=E\left[W(t)Z(t+\tau)\right]$

$=E\left\{\left[A(t)+B(t)\right]\left[A(t+\tau)-B(t+\tau)\right]\right\}$

$=E\left[A(t)A(t+\tau)-A(t)B(t+\tau)+B(t)A(t+\tau)-B(t)B(t+\tau)\right]$

$=E\left[A(t)A(t+\tau)\right]-E\left[A(t)B(t+\tau)\right]+E\left[B(t)A(t+\tau)\right]-E\left[B(t)B(t+\tau)\right]$

$=R_{AA}(\tau)-R_{AB}(\tau)+R_{BA}(\tau)-R_{BB}(\tau)$

$=R_{AA}(\tau)-R_{BB}(\tau)$

$=e^{-|\tau|}-3e^{-|\tau|}$

$R_{WZ}(\tau)=-2e^{-|\tau|}$

b). The random processes $\{X(t)\}$ and $\{Y(t)\}$ defined by $X(t)=A\cos\omega t+B\sin\omega t, Y(t)=B\cos\omega t-A\sin\omega t$ where A & B are uncorrelated zero mean random variables with same variance find its autocorrelation function.

Solution:

Given $E(A)=E(B)=0$ & $E\left(A^2\right)=E\left(B^2\right)=\sigma^2$

Also A and B are uncorrelated i.e., $E(AB)=0$

$R_{XY}(\tau)=E\left[X(t)Y(t+\tau)\right]$

$=E\left\{\left[A\cos\omega t+B\sin\omega t\right]\left[B\cos\omega(t+\tau)-A\sin\omega(t+\tau)\right]\right\}$

$=E\begin{bmatrix}AB\cos\omega t\cos\omega(t+\tau)-A^2\cos\omega t\sin\omega(t+\tau)\\+B^2\sin\omega t\cos\omega(t+\tau)-BA\sin\omega t\sin\omega(t+\tau)\end{bmatrix}$

$=E(AB)\cos\omega t\cos\omega(t+\tau)-E\left(A^2\right)\cos\omega t\sin\omega(t+\tau)$

$+E\left(B^2\right)\sin\omega t\cos\omega(t+\tau)-E(BA)\sin\omega t\sin\omega(t+\tau)$

$=E\left(B^2\right)\sin\omega t\cos\omega(t+\tau)-E\left(A^2\right)\cos\omega t\sin\omega(t+\tau)\left(\because E(AB)=0\right)$

$=\sigma^2\left[\sin\omega t\cos\omega(t+\tau)-\cos\omega t\sin\omega(t+\tau)\right]\left[\because E\left(A^2\right)=E\left(B^2\right)=\sigma^2\right]$

$=\sigma^2\sin\omega(t-t-\tau)$

$=\sigma^2\sin\omega(-\tau)$

$R_{XY}(\tau)=-\sigma^2\sin\omega t$ $\left[\because \sin(-\theta)=-\sin\theta\right]$

Problem 22. a). Consider two random processes $X(t) = 3\cos(\omega t + \theta)$ and $Y(t) = 2\cos\left(\omega t + \theta - \frac{\pi}{2}\right)$ where θ is a random variable uniformly distributed in $(\theta, 2\pi)$. Prove that $\sqrt{R_{XX}(0)\,R_{YY}(0)} \geq |R_{XY}(\tau)|$.

Solution:

Given $X(t) = 3\cos(\omega t + \theta)$

$Y(t) = 2\cos\left(\omega t + \theta - \frac{\pi}{2}\right)$

$$R_{XX}(\tau) = E[X(t)X(t+\tau)]$$

$$= E[3\cos(\omega t + \theta)\,3\cos(\omega t + \omega\tau + \theta)]$$

$$= \frac{9}{2}E[\cos(2\omega t + \omega\tau + 2\theta) + \cos(-\omega\tau)]$$

$\because$ θ is uniformly distributed in $(0, 2\pi)$, $f(\theta) = \frac{1}{2\pi}$

$$= \frac{9}{2}\int_0^{2\pi}[\cos(2\omega t + \omega\tau + 2\theta) + \cos\omega\tau]\frac{1}{2\pi}\,d\theta$$

$$= \frac{9}{2}\frac{1}{2\pi}\left[\frac{\sin(2\omega t + \omega\tau + 2\theta)}{2} + \theta\cos\omega\tau\right]_0^{2\pi}$$

$$= \frac{9}{4\pi}[0 + 2\pi\cos\omega t] \quad \left[\because \left[\frac{\sin(2\omega t + \omega\tau + 2\theta)}{2}\right]_0^{2\pi} = 0\right]$$

$$R_{XX}(\tau) = \frac{9}{2}\cos\omega\tau$$

$$R_{XX}(0) = \frac{9}{2}$$

$$R_{YY}(\tau) = E[Y(t)Y(t+\tau)]$$

$$= E\left[2\cos\left(\omega t + \theta - \frac{\pi}{2}\right)2\cos\left(\omega t + \omega\tau + \theta - \frac{\pi}{2}\right)\right]$$

$$= \frac{4}{2}E[\cos(2\omega t + \omega\tau + 2\theta - \pi) + \cos\omega\tau]$$

$$= 2\int_0^{2\pi}[\cos(2\omega\tau + \omega\tau + 2\theta - \pi) + \cos\omega\tau]\frac{1}{2\pi}\,d\theta$$

$$= \frac{1}{\pi}\left[\frac{\sin(2\omega t + \omega\tau + 2\theta - \pi)}{2} + \theta\cos\omega\tau\right]_0^{2\pi}$$

$$= 2\cos\omega\tau$$

Similarly, $R_{XY}(\tau) = 3\cos\left(\omega\tau - \frac{\pi}{2}\right)$

$$\left|R_{XY}(\tau)\right|_{\max} = 3$$

By property (2) i.e., $\sqrt{R_{XX}(0)R_{YY}(0)} > \left|R_{XY}(\tau)\right|$

$$\therefore \sqrt{\frac{9}{2}.2} \geq 3\cos\left(\omega\tau - \frac{\pi}{2}\right), \forall\tau$$

In this care $R_{XY}(\tau)$ takes on its maximum possible value of 3 at $\tau = \frac{\pi}{2\omega} + \frac{n2\pi}{\omega}, \forall n$

Since it is periodic

$$R_{XX}(0)R_{YY}(0) = \left|R_{XY}(\tau)\right|^2_{\max}$$

b). State and prove Wiener – Khinchine Theorem

Solution:

Statement:

If $X_T(\omega)$ is the Fourier transform of the truncated random process defined as

$$X_T(t) = \begin{cases} X(t), & -T \leq t \leq T \\ 0, & otherwise \end{cases}$$

Where $\{X(t)\}$ is a real WSS process with power spectral function $S_{XX}(\omega)$, then

$$S_{XX}(\omega) = \lim_{T\to\infty}\left[\frac{1}{2T}E\left\{\left|X_T(\omega)\right|^2\right\}\right]$$

Proof:

Given $X_T(\omega) = \int_{-\infty}^{\infty} X_T(t)\, e^{-i\omega t}dt$

$$= \int_{-T}^{T} X(t)\, e^{-i\omega t}dt$$

Now $\left|X_T(\omega)\right|^2 = X_T(\omega)X_T(-\omega)$

$$= \int_{-T}^{T} X(t_1)e^{-i\omega t_1}dt_1 \int_{-T}^{T} X(t_2)e^{-i\omega t_2}dt_2$$

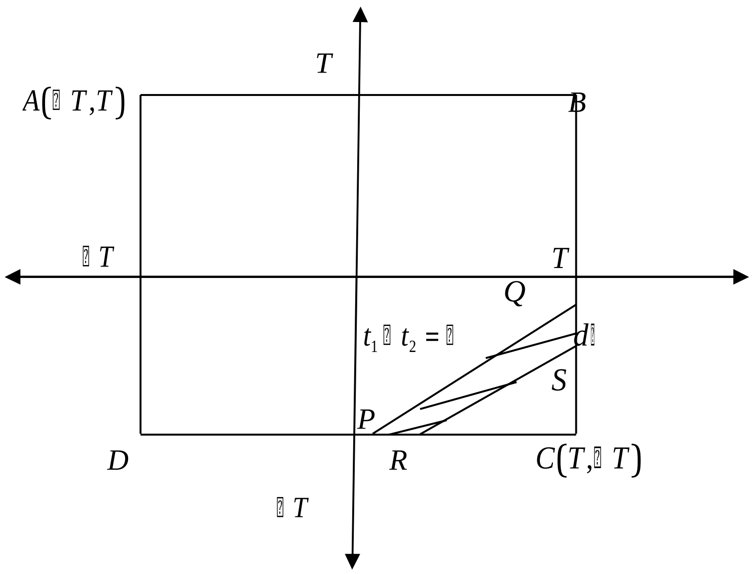
$$= \int_{-T}^{T}\int_{-T}^{T} X(t_1)X(t_2)e^{-i\omega(t_1-t_2)}dt_1\,dt_2$$

$$\therefore E\left\{\left|X_T(\omega)\right|^2\right\} = \int_{-T}^{T}\int_{-T}^{T} E\left[X(t_1)X(t_2)\right]e^{-i\omega(t_1-t_2)}dt_1\,dt_2$$

$$= \int_{-T}^{T}\int_{-T}^{T} R(t_1-t_2)e^{-i\omega(t_1-t_2)}dt_1\,dt_2$$

$$= \int_{-T}^{T}\int_{-T}^{T} \varnothing(t_1 - t_2)\, dt_1 dt_2 \text{-----------------} (1)$$

$\because$ $\{X(t)\}$ is WSS its autocorrelation is a function of time difference ie., $t_1 - t_2 = \tau$



The double integral (1) is evaluated over the area of the square ABCD bounded by the $t_1 = -T, T$ & $t_2 = -T, T$ as shown in the figure

We divide the area of the square into a number of strips like PQRS, where PQ is given by $t_1 - t_2 = \tau$ and RS is given by $t_1 - t_2 = \tau + d\tau$.

When PQRS is at A, $t_1 - t_2 = -T - T = -2T$ and $t_1 - t_2 = 2T$ and at C

$\therefore \tau$ varies from $-2T$ to $2T$ such that the area ABCD is covered.

Now $dt_1 dt_2$ = elemental area of the t_1, t_2 plane

=Area of the strip PQRS (2)

At P, $t_2 = -T$, $t_1 = \tau + t_2 = \tau - T$ At

Q, $t_1 = T$, $t_2 = t_1 - \tau = T - \tau$ $\therefore$ $PC =$

$CQ = 2T - \tau, \tau > 0$

$PC = CQ = 2T + \tau, \tau < 0$

$RC = SC = 2T - \tau - d\tau, \tau > 0$

When $\tau > 0$,

Area of PQRS = $\Delta PCQ - \Delta RCS$

$$= \frac{1}{2}(2T - \tau)^2 - \frac{1}{2}(2T - \tau - d\tau)^2$$

$$= (2T - \tau)\, d\tau \quad \text{Omitting } (d\tau)^2 \text{ ------------------------ } (3)$$

From (2) & (3)

$dt_1 dt_2 = \left(2T - |\tau|\, d\tau\right) - - - - - (4)$

Using (4) in (1)

$$E\left\{|X_T(\omega)|^2\right\} = \int_{-2T}^{2T} \phi(\tau)\left(2T - |\tau|\right)d\tau$$

$$\frac{1}{2T} E\left\{|X_T(\omega)|^2\right\} = \int_{-2T}^{2T} \phi(\tau)\left(1 - \frac{|\tau|}{2T}\right)d\tau$$

$$\therefore \lim_{T\to\infty} \frac{1}{2T} E\left\{|X_T(w)|^2\right\} = \lim_{T\to\infty} \int_{-2T}^{2T} \phi(\tau)\, d\tau - \lim_{T\to\infty} \int \frac{|\tau|}{2T} d\tau$$

$$= \int_{-\infty}^{\infty} \phi(\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau$$

$$= S(\omega)$$

This theorem provides an alternative method for finding $S(\omega)$ for a WSS Process.

Problem 23. a). Define the spectral density $S(\omega)$ of a real valued stochastic processes $\{X(t) : t \geq 0\}$, obtain the spectral density of $\{Y(t) : t \geq 0\}, Y(t) = \alpha X(t)$ when α is increment of $X(t)$ such that $P(\alpha = 1) = P(\alpha = -1) = \frac{1}{2}$.

Solution:

If $\{X(t)\}$ is a stationary process {either in the strict serve or wide sense with autocorrelation function $R(\tau)$, then the Fourier transform of $R(\tau)$ is called the Power spectral density function of $\{X(t)\}$ denoted by $S(w)$

Thus $S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau$

Given $P\{\alpha = 1\} = P\{\alpha = -1\} = \frac{1}{2}$

$\therefore E(\alpha) = 1.\frac{1}{2} + (-1).\frac{1}{2} = 0$

$\therefore E(\alpha^2) = (1)^2.\frac{1}{2} + (-1)^2.\frac{1}{2} = 1$

$E[Y(t)] = E[\alpha X(t)]$

$= E(\alpha) E[X(t)]$ $[\because \alpha\ X(t)\ are\ independent]$

$E[Y(t)] = 0$ $[\because E(\alpha) = 0]$

$R_{YY}(\tau) = E[Y(t)Y(t+\tau)]$

$= E[\alpha X(t) \alpha X(t+\tau)]$

$$= E\left[\alpha^2 X(t) X(t+\tau)\right]$$
$$= E\left[\alpha^2\right] E\left[X(t) X(t+\tau)\right]$$
$$= 1\, E\left[X(t) X(t+\tau)\right]$$
$$= (1) P\left[X(t)(t+\tau) = 1\right] + (-1) P\left[X(t)(t+\tau) = -1\right]$$
$$= (1)\sum_{n=even} \frac{e^{-\lambda T}(\lambda T)}{n!} + (-1)\sum_{n=odd} \frac{e^{-\lambda T}(\lambda T)^n}{n!}$$
$$= e^{-\lambda T} cosh\lambda\tau - e^{-\lambda T} sinh\lambda\tau$$
$$= e^{-\lambda\tau}\left[cosh\lambda T - sinh\lambda T\right]$$
$$= e^{-\lambda T} e^{-\lambda T} = e^{-2\lambda T}$$

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau$$
$$= \int_{-\infty}^{\infty} e^{-2\lambda|\tau|} e^{-i\omega\tau} d\tau$$
$$= \int_{-\infty}^{0} e^{2\lambda\tau} e^{-i\omega\tau} d\tau + \int_{0}^{\infty} e^{-2\lambda\tau} e^{-i\omega\tau} d\tau$$
$$= \int_{-\infty}^{0} e^{(2\lambda - i\omega)\tau} d\tau + \int_{0}^{\infty} e^{-(2\lambda + i\omega)\tau} d\tau$$
$$= \left[\frac{e^{(2\lambda - i\omega)\tau}}{2\lambda - i\omega}\right]_{-\infty}^{0} + \left[\frac{e^{-(2\lambda + i\omega)\tau}}{-(2\lambda + i\omega)}\right]_{0}^{\infty}$$
$$= \frac{1}{2\lambda - i\omega} + \frac{1}{2\lambda + i\omega}$$
$$= \frac{2\lambda - i\omega + 2\lambda - i\omega}{4\lambda^2 - \omega^2}$$
$$= \frac{4\lambda}{4\lambda^2 + \omega^2}$$

b). Show that (i) $S(\omega) \geq 0$ & (ii) $S(\omega) = S(-\omega)$ where $S(\omega)$ is the spectral density of a real valued stochastic process.

Solution:

(i) $S(\omega) \geq 0$

Proof:

If possible let $S(\omega) < 0$ at $\omega = \omega_0$

i.e., let $S(\omega) < 0$ in $\omega_0 - \frac{\varepsilon}{2} < \omega < \omega_0 + \frac{\varepsilon}{2}$, where ε is very small

Let us assume that the system function of the convolution type linear system is

$$H(\omega) = \begin{cases} 1 & , \omega_0 - \frac{\varepsilon}{2} < \omega < \omega_0 + \frac{\varepsilon}{2} \\ 0 & , elsewhere \end{cases}$$

Now $S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$

$$= \begin{cases} S_{XX}(\omega), & \omega_0 - \frac{\varepsilon}{2} < \omega < \omega_0 + \frac{\varepsilon}{2} \\ 0 & , elsewhere \end{cases}$$

$$E\left[Y^2(t)\right] = R_{YY}(0)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)\,d\omega$$

$$= \frac{1}{2\pi}\int_{\omega_0-\frac{\varepsilon}{2}}^{\omega_0+\frac{\varepsilon}{2}} S_{XX}(\omega)\,d\omega$$

$$= \frac{\varepsilon}{2\pi} S_{XX}(\omega_0)$$

[$\because$ $S_{XX}(\omega)$ Considered a constant $S_{XX}(\omega_0)$ band is narrow] $\therefore$ $E\left[Y^2(t)\right] \geq 0$, $S_{XX}(\omega_0) \geq 0$ which is contrary to our initial assumption.

$\therefore$ $S_{XX}(\omega) \geq 0$ everywhere, since $\omega = \omega_0$ is arbitrary

(ii) $S(\omega) = S(-\omega)$

Proof:

Consider $S(-\omega) = \int_{-\infty}^{\infty} R(\tau) e^{i\omega\tau} d\tau$

Let $\tau = -u$ then $d\tau = -du$ and u varies from ∞ and $-\infty$

$$S(-\omega) = \int_{\infty}^{-\infty} R(-u) e^{-i\omega u} (-du)$$

$$= \int_{-\infty}^{\infty} R(-u) e^{-i\omega u} du$$

$$= \int_{-\infty}^{\infty} R(u) e^{-i\omega u} du \quad \left[\because R(\tau) \text{ is an even function of } \tau\right]$$

$$= S(\omega)$$

Hence $S(\omega)$ is even function of ω

Problem 24. a). An autocorrelation function R (τ) of $\{X(t); -\infty < t < \infty\}$ in given by $ce^{-\alpha|\tau|}, c > 0, \alpha > 0$. Obtain the spectral density of R (τ).

Solution:

Given $R(\tau) = ce^{-\alpha|\tau|}$

By definition

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} c e^{-\alpha|\tau|} e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{0} c e^{-\alpha(-\tau)} e^{-i\omega\tau} d\tau + \int_{0}^{\infty} c e^{-\alpha(\tau)} e^{-i\omega\tau} d\tau$$

$$= c\left[\int_{-\infty}^{0} e^{(\alpha-i\omega)\tau} d\tau + \int_{0}^{\infty} e^{-(\alpha-i\omega)\tau} d\tau\right]$$

$$= c\left\{\left[\frac{e^{(\alpha-i\omega)\tau}}{\alpha - i\omega}\right]_{-\infty}^{0} + \left[\frac{e^{-(\alpha+i\omega)\tau}}{-(\alpha+i\omega)}\right]_{0}^{\infty}\right\}$$

$$= c\left[\frac{1}{\alpha - i\omega} + \frac{1}{\alpha + i\omega}\right]$$

$$= c\left[\frac{\alpha + i\omega + \alpha - i\omega}{\alpha^2 - \omega^2}\right]$$

$$S(\omega) = \frac{2\alpha c}{\alpha^2 + \omega^2}$$

b). Given that a process $\{X(t)\}$ has the autocorrelation function $R_{XX}(\tau) = Ae^{-\alpha|\tau|}\cos(\omega_0\tau)$ where $A > 0$, $\alpha > 0$ and ω_0 are real constants, find the power spectrum of $X(t)$.

Solution:

By definition $S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau$

$$= \int_{-\infty}^{\infty} Ae^{-\alpha|\tau|} \cos(\omega_0\tau) e^{-i\omega\tau} d\tau$$

$$= A\int_{-\infty}^{\infty} e^{-\alpha|\tau|} \cos(\omega_0\tau) \left[\cos\omega\tau - i\sin\omega\tau\right] d\tau$$

$$= A\int_{-\infty}^{\infty} e^{-\alpha|\tau|} \cos\omega_0\tau \cos\omega\tau + A\int_{-\infty}^{\infty} e^{-\alpha|\tau|} \cos\omega_0\tau \left(-\sin\omega\tau\right) d\tau$$

$$= 2A\int_{0}^{\infty} e^{-\alpha\tau} \cos\omega_0\tau \cos\omega\tau \, d\tau$$

$$= 2A\int_{-\infty}^{\infty} e^{-\alpha\tau} \left[\frac{\cos(\omega_0+\omega)\tau + \cos(\omega-\omega_0)\tau}{2}\right] d\tau$$

Using $\int_{0}^{\infty} e^{-ax} \cos bx \, dx = \dfrac{a}{a^2 + b^2}$

$$S(\omega) = A\left[\frac{\alpha}{\alpha^2+(\omega_0+\omega)^2}+\frac{\alpha}{\alpha^2+(\omega_0-\omega)^2}\right]$$

$$= A\alpha\left[\frac{1}{\alpha^2+(\omega_0+\omega)^2}+\frac{1}{\alpha^2+(\omega_0-\omega)^2}\right]$$

Problem 25. a). Find the power spectral density of the random binary transmission process whose autocorrelation function is $R(\tau) = \begin{cases} 1-|\tau| & \text{for } |\tau| \le 1 \\ 0 & \text{elsewhere} \end{cases}$.

Solution:

By definition $S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-i\omega\tau}d\tau$

$$= \int_{-1}^{1}\left[1-|\tau|\right]e^{-i\omega\tau}d\tau$$

$$= \int_{-1}^{0}\left[1+\tau\right]e^{-i\omega\tau}d\tau + \int_{0}^{1}\left[1-\tau\right]e^{-i\omega\tau}d\tau$$

$$= \int_{-1}^{0}\left[1+\tau\right]e^{-i\omega\tau}d\tau + \int_{0}^{1}\left[1-\tau\right]e^{-i\omega\tau}d\tau$$

$$= \left[(1+\tau)\frac{e^{-i\omega\tau}}{-i\omega} - \frac{e^{-i\omega\tau}}{i^2\omega^2}\right]_{-1}^{0} + \left[(1-\tau)\frac{e^{-i\omega\tau}}{-i\omega} - (-1)\frac{e^{-i\omega\tau}}{i^2\omega^2}\right]_{0}^{1}$$

$$= \left[\frac{2}{\omega^2} - \frac{e^{-i\omega} - e^{-i\omega}}{\omega^2}\right]$$

$$= \frac{1}{\omega^2}\left[2-2cos\omega\right]$$

$$= \frac{2}{\omega^2}\left(1-cos\omega\right)$$

$$= \frac{2}{\omega^2}\left[1-\left(1-sin^2\frac{\omega}{2}\right)\right]$$

$$= \frac{2}{\omega^2}\left[2sin^2\frac{\omega}{2}\right]$$

$$= \frac{4}{\omega^2}sin\frac{\omega}{2}$$

$$S(\omega) = \left[\frac{sin\frac{\omega}{2}}{\frac{\omega}{2}}\right]^2$$

b). Show that the power spectrum of the autocorrelation function $e^{-\alpha|\tau|}\left[1+\alpha|\tau|\right]$ is $\frac{4\alpha^3}{\left(\alpha^2+w^2\right)^2}$.

Solution:

By definition, $S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-i\omega\tau}d\tau$

$$= \int_{-\infty}^{\infty} e^{-\alpha|\tau|}\left[1+\alpha|\tau|\right]e^{-i\omega\tau}d\tau$$

$$= \int_{-\infty}^{0} e^{(\alpha-i\omega)\tau}(1-\alpha\tau)\,d\tau + \int_{0}^{\infty} e^{-(\alpha+i\omega)\tau}(1+\alpha\tau)\,d\tau$$

$$= \left[(1-\alpha\tau)\frac{e^{(\alpha-i\omega)\tau}}{\alpha-i\omega} - (-\alpha)\frac{e^{(\alpha-i\omega)\tau}}{(\alpha-i\omega)^2}\right]_{-\infty}^{0} + \left[(1+\alpha\tau)\frac{e^{(\alpha+i\omega)\tau}}{-(\alpha+i\omega)} - (\alpha)\frac{e^{-(\alpha+i\omega)\tau}}{(\alpha+i\omega)^2}\right]_{0}^{\infty}$$

$$= \left[\frac{1}{\alpha-i\omega} + \frac{\alpha}{(\alpha-i\omega)^2}\right] + \left[0+\frac{1}{\alpha+i\omega} + \frac{\alpha}{(\alpha+i\omega)^2}\right]$$

$$= \left[\frac{1}{\alpha-i\omega} + \frac{1}{\alpha+i\omega}\right] + \alpha\left[\frac{1}{(\alpha-i\omega)^2} + \frac{1}{(\alpha+i\omega)^2}\right]$$

$$= \frac{2\alpha}{\alpha^2+\omega^2} + \alpha\left[\frac{(\alpha+i\omega)^2+(\alpha-i\omega)^2}{\left(\alpha^2+\omega^2\right)^2}\right]$$

$$= \frac{2\alpha}{\alpha^2+\omega^2} + \frac{2\alpha\left(\alpha^2-\omega^2\right)}{\left(\alpha^2+\omega^2\right)^2}$$

$$= \frac{2\alpha\left(\alpha^2+\omega^2\right)+2\alpha\left(\alpha^2-\omega^2\right)}{\left(\alpha^2+\omega^2\right)^2}$$

$$= \frac{2\alpha\left(\alpha^2+\omega^2+\alpha^2-\omega^2\right)}{\left(\alpha^2+\omega^2\right)^2}$$

$$= \frac{4\alpha^3}{\left(\alpha^2-\omega^2\right)^2}$$

Problem 26. a). Find the spectral density function whose autocorrelation function is given by $R(\tau) = \begin{cases} 1-\frac{|\tau|}{\tau} & ; |\tau| \le T \\ 0 & ;\ \text{elsewhere} \end{cases}$

Solution:

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau$$

$$= \int_{-T}^{T} \left(1 - \frac{|\tau|}{\tau}\right) e^{-i\omega\tau} d\tau$$

$$= \int_{-T}^{T} \left(1 - \frac{|\tau|}{T}\right) [\cos\omega\tau - i\sin\omega\tau \; d\tau]$$

$$= \int_{-T}^{T} \left(1 - \frac{|\tau|}{T}\right) \cos\omega\tau \, d\tau$$

$$= 2\int_{0}^{T} \left(1 - \frac{|\tau|}{T}\right) \cos\omega\tau \, d\tau$$

$$= 2\int_{0}^{T} \left(1 - \frac{\tau}{T}\right) \cos\omega\tau \, d\tau$$

$$= 2\left[\left(1 - \frac{\tau}{T}\right)\left(\frac{\sin\omega\tau}{\omega}\right) - \left(\frac{-1}{\tau}\right)\left(\frac{-\cos\omega\tau}{\omega^2}\right)\right]_0^T$$

$$= 2\left[\frac{-\cos\omega T}{\omega^2 T} + \frac{1}{\omega^2 T}\right]$$

$$= \frac{2}{\omega^2 T}[1 - \cos\omega T]$$

$$= \frac{2}{\omega^2 T} 2\sin^2\left(\frac{\omega T}{2}\right)$$

$$S(\omega) = \frac{4\sin^2\left(\frac{\omega T}{2}\right)}{\omega^2 T}$$

b). Find the power spectral density function whose autocorrelation function is given by $R(\tau) = \frac{A^2}{2}\cos(\omega_0\tau)$

Solution:

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} \frac{A^2}{2}\cos(\omega_0\tau) e^{-i\omega\tau} d\tau$$

$$= \frac{A^2}{2}\int_{-\infty}^{\infty} \cos(\omega_0\tau)[\cos\omega\tau - i\sin\omega\tau] d\tau$$

$$= \frac{A^2}{4}\left[\int_{-\infty}^{\infty}[\cos(\omega-\omega_0)\tau + \cos(\omega+\omega_0)\tau] d\tau - i\int_{-\infty}^{\infty}[\sin(\omega+\omega_0)\tau + \sin(\omega-\omega_0)] d\tau\right]$$

$$= \frac{A^2}{4}\left[\int_{-\infty}^{\infty}\left[\cos(\omega+\omega_0)\tau - i\sin(\omega+\omega_0)\tau\right]d\tau + \int_{-\infty}^{\infty}\left[\cos(\omega-\omega_0)\tau - i\sin(\omega-\omega_0)\tau\right]d\tau\right]$$

$$= \frac{A^2}{4}\left[\int_{-\infty}^{\infty} e^{-i(\omega+\omega_0)\tau}\,d\tau + \int_{-\infty}^{\infty} e^{-i(\omega-\omega_0)\tau}\,d\tau\right]$$

By definition of Dirac delta function

$$S(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega\tau}\,d\tau$$

$$2\pi\delta(\omega) = \int_{-\infty}^{\infty} e^{-i\omega\tau}\,d\tau$$

∴ (1) reduces to

$$S(\omega) = \frac{A^2}{4}\left[2\pi\delta(\omega+\omega_0) + 2\pi\delta(\omega-\omega_0)\right]$$

$$S(\omega) = \frac{\pi A^2}{4}\left[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right]$$

Problem 27. a). If the power spectral density of a WSS process is given by $S(\omega) = \begin{cases} \frac{b}{a}(a-|\omega|) & ; |\omega| \le a \\ 0 & ; |\omega| > a \end{cases}$ find its autocorrelation function of the process.

Solution:

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{i\omega\tau}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-a}^{a}\frac{b}{a}(a-|\omega|)e^{i\omega\tau}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-a}^{a}\frac{b}{a}(a-|\omega|)(\cos\omega\tau + i\sin\omega\tau)\,d\omega$$

$$= \frac{2}{2\pi}\int_{0}^{a}\frac{b}{a}(a-\omega)\cos\omega\tau\,d\omega$$

$$= \frac{b}{\pi a}\left[(a-\omega)\frac{\sin\omega\tau}{\tau} - \frac{\cos\omega\tau}{\tau^2}\right]_0^a$$

$$= \frac{b}{\pi a}\left[-\frac{\cos a\tau}{\tau^2} + \frac{1}{\tau^2}\right]$$

$$= \frac{b}{\pi a\tau^2}\left[1-\cos a\tau\right]$$

$$= \frac{b}{\pi a\tau^2}\,2\sin^2\left(\frac{a\tau}{2}\right)$$

b).The power spectral density function of a zero mean WSS process $\{X(t)\}$ is given by $S(\omega) = \begin{cases} 1 \ ; |\omega| < \omega_0 \\ 0 \ ; \text{elsewhere} \end{cases}$. Find $R(\tau)$ and show also that $X(t)$ & $X\left(t + \frac{\pi}{\omega_0}\right)$ are uncorrelated

Solution:

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega) e^{i\omega\tau} d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{i\omega\tau} d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{i\omega\tau}}{i\tau}\right]_{-\omega_0}^{\omega_0}$$

$$= \frac{1}{\pi\tau}\left[\frac{e^{i\omega_0\tau} - e^{-i\omega_0\tau}}{2i}\right]$$

$$= \frac{\sin\omega_0\tau}{\pi\tau}$$

To show that $X(t)$ & $X\left(t + \frac{\pi}{\omega_0}\right)$ are uncorrelated we have to show that the auto covariance is zero.

i.e., $C\left[X(t)X\left(t + \frac{\pi}{\omega_0}\right)\right] = 0$

Consider,

$$C\left[X(t)X\left(t + \frac{\pi}{\omega_0}\right)\right] = R_{XX}\left(\frac{\pi}{\omega_0}\right) - E\left[X(t)\right]E\left[X\left(t + \frac{\pi}{\omega_0}\right)\right]$$

$$= R_{XX}\left(\frac{\pi}{\omega_0}\right) - 0$$

$$C\left[X(t)X\left(t + \frac{\pi}{\omega_0}\right)\right] = \frac{\sin\pi}{\pi\tau} = 0$$

Hence $X(t)$ & $X\left(t + \frac{\pi}{\omega_0}\right)$ are uncorrelated.

Problem 28. a). The power spectrum of a WSS process $\{X(t)\}$ is given by $S(\omega) = \frac{1}{(1+\omega^2)^2}$. Find the autocorrelation function and average power of the process.

Solution:

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega) e^{i\omega\tau} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\left(1+\omega^2\right)^2} e^{i\omega\tau}\, d\omega \quad \text{-------} (1)$$

The integral in (1) is evaluated by contour integration technique as given below.

Consider $\int_C \frac{e^{iaz}dz}{\left(1+z^2\right)}$, where C is the closed contour consisting of the real axis from $-R$ to R and the upper half of the O^{le} , $|Z| = R$

$1+Z^2 = 0$

$Z^2 = -1$

$Z = \pm i$

$Z = i$ is a double pole.

$$[\text{Res}] \quad Z = i = \frac{1}{(z-1)!} \underset{z\to i}{Lt} \frac{d}{dz}\left[(z-i)^2 + (z)\right]$$

$$= \frac{1}{1!} \underset{z\to i}{Lt} \frac{d}{dz}\left[(z-i)^2 \frac{e^{iaz}}{\left(z^2+1\right)^2}\right]$$

$$= \underset{z\to i}{Lt} \frac{d}{dz} \frac{e^{iaz}}{(z+i)^2}$$

$$= \underset{z\to i}{Lt}\left[\frac{(z+i)^2\, i\,a\,e^{iaz} - e^{iaz}\, 2(z+i).1}{(z+i)^4}\right]$$

$$= \underset{z\to i}{Lt}\left[\frac{(z+i)\, i\,a\,e^{iaz} - 2e^{iaz}}{(z+i)^3}\right]$$

$$= \frac{e^{-a}(-2a-2)}{8(-i)} = \frac{2e^{-a}(1+a)}{8i} = \frac{e^{-a}(1+a)}{4i}$$

By Cauchy residue theorem, $\int_c f(z)dz = 2\pi i \sum_{i=1}^{n} \underset{z=a_i}{Res} f(z)$ and taking limits $R \to \infty$ and using Jordan's lemma

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{\left(1+x^2\right)^2} dx = \frac{2\pi i e^{-a}(1+a)}{4i} = \frac{\pi e^{-a}(1+a)}{2} \quad \text{-----} (2)$$

Using (2) in (1)

$$R(\tau) = \frac{1}{2\pi} \frac{(1+\tau)e^{-\tau}\pi}{2} = \frac{(1+\tau)e^{-\tau}}{4}$$

Average power of $\{X(t)\} = \left[R(\tau)\right]_{\tau=0}$

$$= \left[\frac{(1+\tau)e^{-\tau}}{4}\right]_{\tau=0} = \frac{1}{4} = 0.25$$

b). Suppose we are given a cross – power spectrum defined by $S_{XY}(\omega) = \begin{cases} a + \dfrac{jb\omega}{W}, & -W < \omega < W \\ 0 & ,\text{elsewhere} \end{cases}$ where $\omega > 0$, a & b are real constants. Find the cross correlation

Solution:

$$R_{XY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega) e^{j\omega\tau} d\omega$$

$$= \frac{1}{2\pi}\int_{-W}^{W}\left(a + \frac{jb\omega}{W}\right) e^{j\omega\tau} d\omega$$

$$= \frac{a}{2\pi}\left[\frac{e^{j\omega\tau}}{j\tau}\right]_{-W}^{W} + \frac{jb}{2\pi W}\left[\frac{\omega e^{j\omega\tau}}{j\tau} - \frac{e^{j\omega\tau}}{j^2\tau^2}\right]_{-W}^{W}$$

$$= \frac{a}{2\pi}\left[\frac{e^{jW\tau} - e^{-jW\tau}}{j\tau}\right] + \frac{jb}{2\pi W}\left[\frac{We^{jW\tau}}{j\tau} + \frac{e^{jW\tau}}{\tau^2} + \frac{We^{-jW\tau}}{j\tau} - \frac{e^{-jW\tau}}{\tau^2}\right]$$

$$= \frac{a\sin(W\tau)}{\pi\tau} + \frac{bW}{\pi W\tau}\left[\frac{e^{jW\tau} - e^{-jW\tau}}{2}\right] + \frac{jb}{\pi W\tau^2}\left[\frac{e^{jW\tau} - e^{-jW\tau}}{2}\right]$$

$$= \frac{a\sin(W\tau)}{\pi\tau} + \frac{b\cos(W\tau)}{\pi\tau} - \frac{b\sin(W\tau)}{\pi W\tau^2}$$

$$= \frac{1}{\pi\tau^2}\left[\left(a\tau - \frac{b}{W}\right)\sin(W\tau) + b\tau\cos(W\tau)\right]$$

Problem 29. a). The cross-power spectrum of real Random process $\{X(t)\}$ and $\{Y(t)\}$ is given by $S_{XY}(\omega) = \begin{cases} a + jb\omega; & |\omega| < 1 \\ 0 & ;\text{elsewhere} \end{cases}$. Find the cross correlation function.

Solution:

$$R_{XY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega) e^{j\omega\tau} d\omega$$

$$= \frac{1}{2\pi}\int_{-1}^{1}(a + jb\omega) e^{j\omega\tau} d\omega$$

$$= \frac{a}{2\pi}\left[\frac{e^{j\omega\tau}}{j\tau}\right]_{-1}^{1} + \frac{b}{2\pi}\left[\frac{\omega e^{j\omega\tau}}{j\tau} - \frac{e^{j\omega\tau}}{j^2\tau^2}\right]_{-1}^{1}$$

$$= \frac{a}{2\pi}\left[\frac{e^{j\tau} - e^{-j\tau}}{j\tau}\right] + \frac{jb}{2\pi}\left[\frac{e^{j\tau}}{j\tau} + \frac{e^{j\tau}}{\tau^2} + \frac{e^{-j\tau}}{j\tau} - \frac{e^{-j\tau}}{\tau^2}\right]$$

$$= \frac{a}{\pi\tau}[\sin\tau] + \frac{jb}{2\pi}\left[\frac{e^{j\tau} - e^{-j\tau}}{j\tau}\right] + \frac{jb}{2\pi}\left[\frac{e^{j\tau} - e^{-j\tau}}{\tau^2}\right]$$

$$= \frac{a\sin\tau}{\pi\tau} + \frac{b\cos\tau}{\pi\tau} - \frac{b\sin\tau}{\pi\tau^2}$$

$$= \frac{1}{\pi\tau^2}\left[\left(a\tau - b\right)\sin\tau + b\tau\cos\tau\right]$$

b). Find the power spectral density function of a WSS process with autocorrelation function. $R(\tau) = e^{-a\tau^2}$.

Solution:

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-i\omega\tau}d\tau$$

$$= \int_{-\infty}^{\infty} e^{-a\tau^2}e^{-i\omega\tau}d\tau$$

$$= \int_{-\infty}^{\infty} e^{-\left(a\tau^2 + i\omega\tau\right)}d\tau$$

$$= \int_{-\infty}^{\infty} e^{-a\left(\tau^2 + \frac{i\omega\tau}{a}\right)}d\tau$$

$$= \int_{-\infty}^{\infty} e^{-a\left(\tau^2 + 2.\frac{i\omega}{a}.\tau\right)}d\tau$$

$$= \int_{-\infty}^{\infty} e^{-a\left[\left(\tau^2 + \frac{2i\omega}{2a}t + \frac{i^2\omega^2}{2^2a^2}\right) - \frac{i^2\omega^2}{2^2a^2}\right]}d\tau$$

$$= \int_{-\infty}^{\infty} e^{-a\left[\left(\tau + \frac{i\omega}{2a}\right)^2 + \frac{\omega^2}{4a^2}\right]}d\tau$$

$$= \int_{-\infty}^{\infty} e^{-a\left[\left(\tau + \frac{i\omega}{2a}\right)^2\right]}.e^{-\frac{\omega^2}{4a^2}}d\tau$$

$$\therefore S_{XX}(\omega) = e^{-\frac{\omega^2}{4a}}\int_{-\infty}^{\infty} e^{-a\left(\tau + \frac{i\omega}{2a}\right)^2}d\tau$$

Now let, $a\left(\tau + \frac{i\omega}{2a}\right)^2 = u^2$ *i.e.* $u = \sqrt{a}\left(\tau + \frac{i\omega}{2a}\right)$

$\therefore \quad du = \sqrt{a}\ d\tau$

and as, $\tau \to -\infty, u \to -\infty$ and as $\tau \to \infty, u \to \infty$

$$\therefore S_{XX}(\omega) = e^{-\frac{\omega^2}{4a}}\int_{-\infty}^{\infty} e^{-u^2}\frac{du}{\sqrt{a}}$$

$$= \frac{e^{-\frac{\omega^2}{4a}}}{\sqrt{a}}.2\int_0^{\infty} e^{-u^2}du \qquad \text{....} \because e^{-u^2} \text{ is even}$$

$$= \frac{2e^{\frac{-\omega^2}{4a}}}{\sqrt{a}}\frac{\sqrt{\pi}}{2} \qquad \text{...} \because \text{Standard Result : } \int_0^{\infty} e^{-X^2}dx = \frac{\sqrt{x}}{2}$$

$$\therefore \qquad S_{XX}(\omega) = \sqrt{\frac{\pi}{a}}\, e^{\frac{-\omega^2}{4a}}$$

Problem 30. a). The autocorrelation function of the random telegraph signal process is given by $R(\tau) = a^2 e^{-2\gamma|\tau|}$. Determine the power density spectrum of the random telegraph signal.

Solution:

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau = \int_{-\infty}^{\infty} a^2 e^{-2\gamma|\tau|} e^{-i\omega\tau} d\tau$$

$$= a^2 \int_{-\infty}^{0} e^{-2\gamma|\tau|} e^{-i\omega\tau} d\tau + a^2 \int_{0}^{\infty} e^{-2\gamma|\tau|} e^{-i\omega\tau} d\tau$$

$$= a^2 \int_{-\infty}^{0} e^{-2\gamma(-\tau)} e^{-i\omega\tau} d\tau + a^2 \int_{0}^{\infty} e^{-2\gamma(\tau)}.e^{-i\omega\tau} d\tau$$

$$= a^2 \int_{-\infty}^{0} e^{(2\gamma - i\omega)\tau} d\tau + a^2 \int_{0}^{\infty} e^{-(2\gamma + i\omega)\tau} d\tau$$

$$= a^2 \left[\frac{e^{(2\gamma - i\omega)\tau}}{(2\gamma - i\omega)}\right]_{-\infty}^{0} + a^2 \left[\frac{e^{-(2\gamma - i\omega)\tau}}{-(2\gamma - i\omega)}\right]_{0}^{\infty}$$

$$= \frac{a^2}{(2\gamma - i\omega)}\left(e^0 - e^{-\infty}\right) - \frac{a^2}{(2\gamma + i\omega)}\left(e^{-\infty} - e^0\right)$$

$$= \frac{a^2}{2\gamma - i\omega}(1 - 0) - \frac{a^2}{(2\gamma + i\omega)}(0 - 1)$$

$$= a^2 \left[\frac{1}{2\gamma - i\omega} + \frac{1}{2\gamma + i\omega}\right] = a^2 \left[\frac{2\gamma + i\omega + 2\gamma - i\omega}{(2\gamma)^2 + \omega^2}\right]$$

i.e., $$S_{XX}(\omega) = a^2 \left[\frac{4\gamma}{4\gamma^2 + \omega^2}\right] = \frac{4a^2\gamma}{4\gamma^2 + \omega^2}$$

b). Find the power spectral density of the random process $\{x(t)\}$ if $E\{x(t)\} = 1$ and $R_{xx}(\tau) = 1 + e^{-\alpha|\tau|}$.

Solution:

$$S(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} \left[1 + e^{-\alpha|\tau|}\right] e^{-i\omega\tau} d\tau = \int_{-\infty}^{\infty} e^{-i\omega\tau} d\tau + \int_{-\infty}^{0} e^{\alpha\tau} e^{-i\omega\tau} d\tau + \int_{0}^{\infty} e^{-\alpha\tau} e^{-i\omega\tau} d\tau$$

$$= \delta(\omega) + \frac{1}{\alpha - i\omega} + \frac{1}{\alpha + i\omega}$$

$$= \delta(\omega) + \frac{2\alpha}{\alpha^2 + \omega^2}$$

****

UNIT – V LINEAR SYSTEMS WITH RANDOM INPUTS

PART - A

Problem 1. If the system function of a convolution type of linear system is given by $h(t) = \begin{cases} \frac{1}{2a} & for\ |t| \le a \\ 0 & for\ |t| > a \end{cases}$ find the relation between power spectrum density function of the input and output processes.

Solution:

$$H(\omega) = \int_{-a}^{a} h(t) e^{-i\omega t} dt = \frac{\sin a\omega}{a\omega}$$

We know that $S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$

$$\Rightarrow S_{YY}(\omega) = \frac{\sin^2 a\omega}{a^2\omega^2} S_{XX}(\omega).$$

Problem 2. Give an example of cross-spectral density.

Solution:

The cross-spectral density of two processes $X(t)$ and $Y(t)$ is given by

$$S_{XY}(\omega) = \begin{cases} p + iq\omega, & if\ |\omega| < 1 \\ 0, & otherwise \end{cases}$$

Problem 3. If a random process $X(t)$ is defined as $X(t) = \begin{cases} A, & 0 \le t \le 1 \\ 0, & otherwise \end{cases}$, where A is a random variable uniformly distributed from $-\theta$ to θ. Prove that autocorrelation function of $X(t)$ is $\frac{\theta^2}{3}$.

Solution:

$$R_{XX}(t, t+\tau) = E\left[X(t).X(t+\tau)\right]$$
$$= E\left[A^2\right] \quad \left[\because X(t) \text{ is constant}\right]$$

But A is uniform in $(-\theta, \theta)$

$$\therefore f(\theta) = \frac{1}{2\theta}, -\theta < a < \theta$$

$$\therefore R_{XX}(t, t+\tau) = \int_{-\theta}^{\theta} a^2 f(a)\, da$$
$$= \int_{-\theta}^{\theta} a^2 . \frac{1}{2\theta} d\theta = \frac{1}{2\theta}\left[\frac{a^3}{3}\right]_{-\theta}^{\theta}$$

$$= \frac{1}{6\theta}\left[\theta^3 - (-\theta)^3\right] = \frac{1}{6\theta}.2\theta^3 = \frac{\theta^2}{3}$$

Problem 4. Check whether $\frac{1}{1+9\tau^2}$ is a valid autocorrelation function of a random process.

Solution: Given $R(\tau) = \frac{1}{1+9\tau^2}$

$$\therefore R(-\tau) = \frac{1}{1+9(-\tau^2)} = \frac{1}{1+9\tau^2} = R(\tau)$$

$\therefore R(\tau)$ is an even function. So it can be the autocorrelation function of a random process.

Problem 5. Find the mean square value of the process X(t) whose power density spectrum is $\frac{4}{4+\omega^2}$.

Solution:

Given $S_{XX}(\omega) = \frac{4}{4+\omega^2}$

Then $R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{i\tau\omega}d\omega$

Mean square value of the process is $E\left[X^2(t)\right] = R_{XX}(0)$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4}{4+\omega^2}d\omega$$

$$= \frac{4}{\pi}\int_0^{\infty} \frac{1}{4+\omega^2}d\omega \qquad \left[\because \frac{1}{4+\omega^2} \text{ is even}\right]$$

$$= \frac{4}{\pi}.\frac{1}{2}\left[\tan^{-1}\frac{\omega}{2}\right]_0^{\infty} = \frac{2}{\pi}\left(\tan^{-1}\infty - \tan^{-1}0\right)$$

$$= \frac{2}{\pi}.\frac{\pi}{2} = 1$$

Problem 6. A Circuit has an impulse response given by $h(t) = \begin{cases} \frac{1}{T} & ; 0 \le t \le T \\ 0 & ; elsewhere \end{cases}$ find the relation between the power spectral density functions of the input and output processes.

Solution:

$$H(\omega) = \int_0^T h(t)\,e^{-i\omega t}dt$$

$$= \int_0^T \frac{1}{T} e^{-i\omega t} dt$$

$$= \frac{1}{T}\left[\frac{e^{-i\omega t}}{-i\omega}\right]_0^T$$

$$= \frac{1}{T}\left[\frac{-e^{-i\omega T}+1}{i\omega}\right]$$

$$= \frac{\left(1-e^{-i\omega T}\right)}{Ti\omega}$$

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$

$$= \frac{\left(1-e^{-i\omega T}\right)^2}{\omega^2 T^2} S_{XX}(\omega)$$

Problem 7. Describe a linear system.

Solution:

Given two stochastic process $\{X_1(t)\}$ and $\{X_2(t)\}$, we say that L is a linear transformation if

$$L\left[a_1 X_1(t) + a_2 X_2(t)\right] = a_1 L\left[X_1(t)\right] + a_2 L\left[X_2(t)\right]$$

Problem 8. Given an example of a linear system.

Solution:

Consider the system f with output $tx(t)$ for an input signal $x(t)$.

i.e. $y(t) = f\left[X(t)\right] = t\,x(t)$

Then the system is linear.

For any two inputs $x_1(t), x_2(t)$ the outputs are $tx_1(t)$ and $tx_2(t)$ Now

$$f\left[a_1 x_1(t) + a_2 x_2(t)\right] = t\left[a_1 x_1(t) + a_2 x_2(t)\right]$$

$$= a_1 t x_1(t) + a_2 t x_2(t)$$

$$= a_1 f\left(x_1(t)\right) + a_2 f\left(x_2(t)\right)$$

$\therefore$the system is linear.

Problem 9. Define a system, when it is called a linear system?

Solution:

Mathematically, a system is a functional relation between input $x(t)$ and output $y(t)$.

Symbolically, $y(t) = f\left[x(t)\right], -\infty < t < \infty$..

The system is said to be linear if for any two inputs $x_1(t)$ and $x_2(t)$ and constants a_1, a_2, $f\left[a_1 x_1(t) + a_2 x_2(t)\right] = a_1 f\left[x_1(t)\right] + a_2 f\left[x_2(t)\right]$.

Problem 10. State the properties of a linear system.

Solution:

Let $X_1(t)$ and $X_2(t)$ be any two processes and a and b be two constants.

If L is a linear filter then

$$L\left[a_1 x_1(t) + a_2 x_2(t)\right] = a_1 L\left[x_1(t)\right] + a_2 L\left[x_2(t)\right].$$

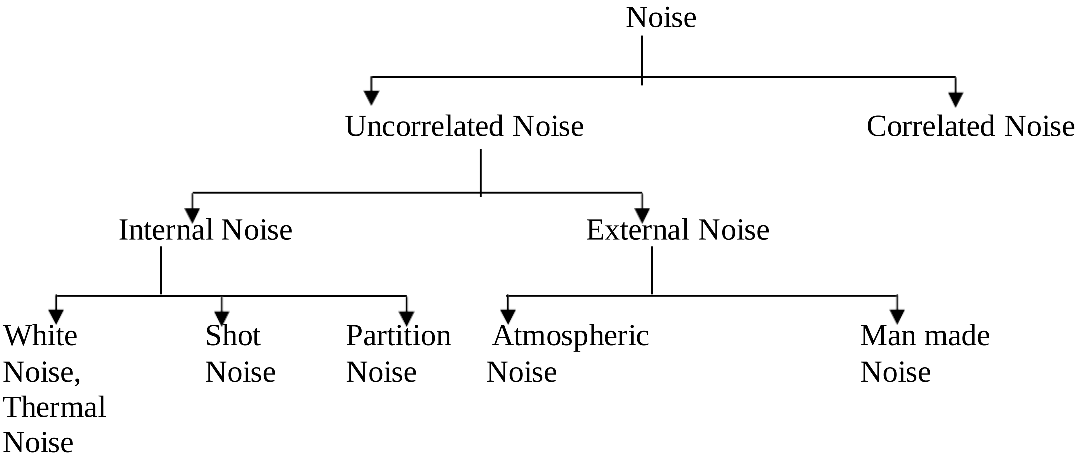
Problem 11. Describe a linear system with an random input.
Solution:

We assume that $X(t)$ represents a sample function of a random process $\{X(t)\}$, the system produces an output or response $Y(t)$ and the ensemble of the output functions forms a random process $\{Y(t)\}$. The process $\{Y(t)\}$ can be considered as the output of the system or transformation f with $\{X(t)\}$ as the input the system is completely specified by the operator f.

Problem 12. State the convolution form of the output of linear time invariant system.
Solution:

If $X(t)$ is the input and $h(t)$ be the system weighting function and $Y(t)$ is the output, then $Y(t) = h(t) * X(t) = \int_{-\infty}^{\infty} h(u) X(t-u)\, du$

Problem 13. Write a note on noise in communication system.
Solution:
The term noise is used to designate unwanted signals that tend to disturb the transmission and processing of signal in communication systems and over which we have incomplete control.



Problem 14. Define band-limited white noise.
Solution:
Noise with non-zero and constant density over a finite frequency band is called band-limit white noise i.e.,

$$S_{NN}(\omega) = \begin{cases} \dfrac{N_0}{2}, & |\omega| \le \omega_B \\ 0, & otherwise \end{cases}$$

Problem 15. Define (a) Thermal Noise (b) White Noise.
Solution:

(a) Thermal Noise: This noise is due to the random motion of free electrons in a conducting medium such as a resistor.

(or)

Thermal noise is the name given to the electrical noise arising from the random motion of electrons in a conductor.

(b) White Noise(or) Gaussian Noise: The noise analysis of communication systems is based on an idealized form of noise called White Noise.

PART-B

Problem 16. A random process $X(t)$ is the input to a linear system whose impulse response is $h(t) = 2e^{-t}, t \geq 0$. If the autocorrelation function of the process is $R_{XX}(\tau) = e^{-2|\tau|}$, find the power spectral density of the output process $Y(t)$.

Solution:

Given $X(t)$ is the input process to the linear system with impulse response $h(t) = 2e^{-t}, t \geq 0$

So the transfer function of the linear system is its Fourier transform

$$H(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-i\omega t} dt$$

$$= \int_{-\infty}^{\infty} 2e^{-t} e^{-i\omega t} dt \qquad \left[\because 2e^{-t}, t \geq 0\right]$$

$$= 2\int_{0}^{\infty} e^{-(1+i\omega)t} dt$$

$$= 2\left[\frac{e^{-(1+i\omega)t}}{-(1+i\omega)}\right]_0^{\infty}$$

$$= \frac{-2}{1+i\omega}[0-1] = \frac{2}{1+i\omega}$$

Given $R_{XX}(\tau) = e^{-2|\tau|}$

$\therefore$ the spectral density of the input is

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} dt$$

$$= \int_{-\infty}^{\infty} e^{-2|\tau|} e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{0} e^{2\tau} e^{-i\omega\tau} d\tau + \int_{0}^{\infty} e^{-2\tau} e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{0} e^{(2-i\omega)\tau} d\tau + \int_{0}^{\infty} e^{-(2+i\omega)\tau} d\tau$$

$$= \left[\frac{e^{(2-i\omega)\tau}}{2-i\omega}\right]_{-\infty}^{0} + \left[\frac{e^{-(2+i\omega)\tau}}{-(2+i\omega)}\right]_{0}^{\infty}$$

$$= \frac{1}{2-i\omega}[1-0] - \frac{1}{2+i\omega}[0-1]$$

$$= \frac{1}{2-i\omega} + \frac{1}{2+i\omega}$$

$$= \frac{2+i\omega+2-i\omega}{(2+i\omega)(2-i\omega)} = \frac{4}{4+\omega^2}$$

We know the power spectral density of the output process $Y(t)$ is given by

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$

$$= \left|\frac{2}{1+i\omega}\right|^2 \frac{4}{4+\omega^2}$$

$$= \frac{4}{(1+\omega^2)} \frac{4}{4+\omega^2}$$

$$= \frac{16}{(1+\omega^2)(4+\omega^2)}$$

Problem 17. If $Y(t) = A\cos(\omega_0 t + \theta) + N(t)$, where A is a constant, θ is a random variable with uniform distribution in $(-\pi, \pi)$ and $N(t)$ is a band-limited Gaussian white noise with a power spectral density $S_{NN}(\omega) = \begin{cases} \frac{N_0}{2}, & \text{for } |\omega - \omega_0| < \omega_B \\ 0, & \text{elsewhere} \end{cases}$. Find the power spectral density of $Y(t)$. Assume that $N(t)$ and θ are independent.

Solution:

Given $Y(t) = A\cos(\omega_0 t + \theta) + N(t)$

$N(t)$ is a band-limited Gaussian white noise process with power spectral density

$$S_{NN}(\omega) = \frac{N_0}{2}, \ |\omega - \omega_0| < \omega_B \quad ie.\ \omega_0 - \omega_B < \omega < \omega_0 + \omega_B$$

Required $S_{YY}(\omega) = \int_{-\infty}^{\infty} R_{YY}(\tau) e^{-i\omega\tau} d\tau$

Now $R_{YY}(\tau) = E[Y(t)Y(t+\tau)]$

$$= E\{[A\cos(\omega_0 t + \theta) + N(t)][A\cos(\omega_0 t + \omega_0\tau + \theta) + N(t+\tau)]\}$$

$$= E\left\{\begin{matrix} A^2\cos(\omega_0 t+\theta)\cos(\omega_0 t+\omega_0\tau+\theta)+N(t)N(t+\tau)+A\cos(\omega_0 t+\theta)N(t+\tau) \\ +A\cos(\omega_0 t+\omega_0\tau+\theta)N(t)\end{matrix}\right\}$$

$$= A^2E\left[\cos(\omega_0 t+\theta).\cos(\omega_0 t+\omega_0\tau+\theta)\right]+E\left[N(t)N(t+\tau)\right]$$

$$+AE\left[\cos(\omega_0 t+\theta)\right]E\left[N(t+\tau)\right]$$

$$+AE\left[\cos(\omega_0 t+\omega_0+\theta)\right]E\left[N(t)\right] \quad [\because \theta \text{ and } N(t) \text{ are independent}]$$

$$= \frac{A^2}{2}\left\{E\left[\cos(2\omega_0 t+\omega_0 t+2\theta)\right]+\cos\omega_0\tau\right\}+R_{NN}(\tau)$$

$$+AE\left[\cos(\omega_0 t+\theta)\right]E\left[N(t+\tau)\right] \quad +AE\left[\cos(\omega_0 t+\omega_0\tau+2\theta)\right]E\left[N(t)\right]\}$$

Since θ is uniformly distributed in $(-\pi,\pi)$ the pdf of θ is $f(\theta)=\frac{1}{2\pi}, -\pi<\theta<\pi$

$$\therefore E\left[\cos(\omega_0 t+\theta)\right]=\int_{-\pi}^{\pi}\cos(\omega_0 t+\theta)\, f(\theta)\, d\theta$$

$$=\int_{-\pi}^{\pi}\left[\cos\omega_0 t.\cos\theta-\sin\omega_0 t.\sin\theta\right]\frac{1}{2\pi}d\theta$$

$$=\frac{1}{2\pi}\left[\cos\omega_0 t\int_{-\pi}^{\pi}\cos\theta\, d\theta-\sin\omega_0 t\int_{-\pi}^{\pi}\sin\theta\, d\theta\right]$$

$$=\frac{1}{2\pi}\left[\cos\omega_0 t\left[\sin\theta\right]_{-\pi}^{\pi}-\sin\omega_0 t.0\right]=0$$

Similarly $E\left[\cos(2\omega_0 t+\omega_0\tau+2\theta)\right]=\int_{-\pi}^{\pi}\cos(2\omega_0 t+\omega_0\tau+2\theta)\frac{1}{2\pi}d\theta$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\cos(2\omega_0 t+\omega_0\tau)\cos 2\theta-\sin(2\omega_0 t+\omega_0\tau)\sin 2\theta\right]d\theta$$

$$=\frac{1}{2\pi}\left\{\cos(2\omega_0 t+\omega_0\tau)\int_{-\pi}^{\pi}\cos 2\theta\, d\theta-\sin(2\omega_0 t+\omega_0\tau)\int_{-\pi}^{\pi}\sin 2\theta\, d\theta\right\}$$

$$=\frac{1}{2\pi}\left\{\cos(2\omega_0 t+\omega_0\tau).\left[\frac{\sin 2\theta}{2}\right]_{-\pi}^{\pi}-\sin(2\omega_0 t+\omega_0\tau).0\right\}=0$$

$$\therefore R_{YY}(\tau)=\frac{A^2}{2}\cos\omega_0\tau+R_{NN}(\tau)$$

$$\therefore S_{YY}(\omega)=\int_{-\infty}^{\infty}\left[\frac{A^2}{2}\cos\omega_0\tau+R_{NN}(\tau)\right]e^{-i\omega\tau}d\tau$$

$$=\frac{A^2}{2}\int_{-\infty}^{\infty}\cos\omega_0\tau\, e^{-i\omega\tau}d\tau+\int_{-\infty}^{\infty}R_{NN}(\tau)e^{-i\omega\tau}d\tau$$

$$=\frac{\pi A^2}{2}\left\{\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right\}+S_{NN}(\omega)$$

$$= \frac{\pi A^2}{2}\left\{\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right\}+\frac{N_0}{2}, \omega_0-\omega_B<\omega<\omega_0+\omega_B$$

Problem 18. Consider a Gaussian white noise of zero mean and power spectral density $\frac{N_0}{2}$ applied to a low pass RC filter whose transfer function is $H(f)=\frac{1}{1+i2\pi fRC}$. Find the autocorrelation function.

Solution:

The transfer function of a RC circuit is given. We know if $X(t)$ is the input process and $Y(t)$ is the output process of a linear system, then the relation between their spectral densities is $S_{YY}(\omega)=|H(\omega)|^2 S_{XX}(\omega)$

The given transfer function is in terms of frequency f

$$S_{YY}(f)=|H(f)|^2 S_{XX}(f)$$

$$S_{YY}(f)=\frac{1}{1+4\pi^2 f^2R^2C^2}\frac{N_0}{2}$$

$$\therefore R_{YY}(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)e^{i\omega\tau}d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i2\pi f}}{1+4\pi^2 f^2R^2C^2}\frac{N_0}{2}df$$

$$=\frac{N_0}{4\pi}\int_{-\infty}^{\infty}\frac{e^{i(2\pi)f}}{4\pi^2R^2C^2\left(\frac{1}{4\pi^2R^2C^2}+f^2\right)}df$$

$$=\frac{N_0}{16\pi^3R^2C^2}\int_{-\infty}^{\infty}\frac{e^{i(2\pi)f}}{\left(\frac{1}{2\pi}\right)^2+f^2}df$$

We know from contour integration $\int_{-\infty}^{\infty}\frac{e^{imx}}{a^2+x^2}dx=\frac{\pi}{a}e^{-m|a|}$

$$\therefore R_{YY}(\tau)=\frac{N_0}{16\pi^3R^2C^2}\frac{\pi}{\frac{1}{2\pi RC}}e^{-\frac{\tau}{2\pi RC}}$$

$$=\frac{N_0}{16\pi^3R^2C^2}2\pi^2RC\,e^{-\frac{\tau}{2\pi RC}}$$

$$=\frac{N_0}{8\pi RC}e^{-\frac{\tau}{2\pi RC}}$$

Problem 19. A wide-sense stationary noise process N(t) has an autocorrelation function $R_{NN}(\tau)=Pe^{-3|\tau|}$, where P is a constant Find its power spectrum

Solution:

Given the autocorrelation function of the noise process $N(t)$ is $R_{NN}(\tau) = Pe^{-3|\tau|}$.

$$\therefore S_{NN}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} P e^{-3|\tau|} e^{-i\omega\tau} d\tau$$

$$= P\left\{ \int_{-\infty}^{0} e^{3\tau} e^{-i\omega\tau} d\tau + \int_{0}^{\infty} e^{-3\tau} e^{-i\omega\tau} d\tau \right\}$$

$$= P\left\{ \int_{-\infty}^{0} e^{(3-i\omega)\tau} d\tau + \int_{0}^{\infty} e^{-(3+i\omega)\tau} d\tau \right\}$$

$$= P\left\{ \left[\frac{e^{(3-i\omega)\tau}}{3-i\omega} \right]_{-\infty}^{0} + \left[\frac{e^{-(3+i\omega)\tau}}{-(3+i\omega)} \right]_{0}^{\infty} \right\}$$

$$= P\left\{ \frac{1}{3-i\omega}(1-0) - \frac{1}{3+i\omega}(0-1) \right\}$$

$$= P\left\{ \frac{1}{3-i\omega} + \frac{1}{3+i\omega} \right\}$$

$$= P\left\{ \frac{3+i\omega+3-i\omega}{(3-i\omega)(3+i\omega)} \right\} = \frac{6P}{9+\omega^2}$$

Problem 20. A wide sense stationary process X(t) is the input to a linear system with impulse response $h(t) = 2e^{-7t}, t \ge 0$. If the autocorrelation function of X(t) is $R_{XX}(\tau) = e^{-4|\tau|}$, find the power spectral density of the output process Y(t).

Solution:

Given X(t) is a WSS process which is the input to a linear system and so the output process Y(t) is also a WSS process (by property autocorrelation function)

Further the spectral relationship is $S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$

Where $S_{XX}(\omega)$ = Fourier transform of $R_{XX}(\tau)$

$$= \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} e^{-4|\tau|} e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{0} e^{4\tau} e^{-i\omega\tau} d\tau + \int_{0}^{\infty} e^{-4\tau} e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{0} e^{\tau(4-i\omega)} d\tau + \int_{0}^{\infty} e^{-\tau(4+i\omega)} d\tau$$

$$= \left[\frac{e^{\tau(4-i\omega)}}{4-i\omega}\right]_{-\infty}^{0} + \left[\frac{e^{-\tau(4+i\omega)}}{4+i\omega}\right]_{0}^{\infty}$$

$$= \frac{1}{4-i\omega}\left[e^{0}-e^{-\infty}\right] + \frac{1}{4+i\omega}\left[e^{-\infty}-e^{0}\right]$$

$$= \frac{1}{4-i\omega} - \frac{1}{4+i\omega}$$

$$S_{XX}(\omega) = \frac{4+i\omega-4+i\omega}{(4-i\omega)(4+i\omega)} = \frac{2i\omega}{16+\omega^2}$$

$H(\omega)$ = Fourier transform of $h(t)$

$$= \int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt$$

$$= \int_{0}^{\infty} 2e^{-7t}.e^{-i\omega t}dt \qquad [\because h(t) = 0 \ if \ t < 0]$$

$$= 2\int_{0}^{\infty} e^{-(7+i\omega)t}dt$$

$$= 2\left[\frac{e^{-(7+i\omega)t}}{-(7+i\omega)}\right]_{0}^{\infty}$$

$$= \frac{-2}{7+i\omega}\left[e^{-\infty}-e^{0}\right] = \frac{2}{7+i\omega}$$

$$\therefore |H(\omega)| = \frac{2}{|7+i\omega|} = \frac{2}{\sqrt{49+\omega^2}}$$

$$\therefore |H(\omega)|^2 = \frac{4}{49+\omega^2}$$

Substituting in (1) we get the power spectral density of Y(t),

$$S_{YY}(\omega) = \frac{4}{49+\omega^2}.\frac{2i\omega}{16+\omega^2} = \frac{8i\omega}{(49+\omega^2)(16+\omega^2)}.$$

Problem 21. A random process X(t) with $R_{XX}(\tau) = e^{-2|\tau|}$ is the input to a linear system whose impulse response is $h(t) = 2e^{-t}, t \geq 0$. Find cross correlation $R_{YY}(\tau)$ between the input process X(t) and the output process Y(t).

Solution:

The cross correlation between input X(t) and output Y(t) to a linear system is

$$R_{XY}(\tau) = R_{XX}(\tau)h(\tau)$$

Taking Fourier transforms, we get

$$S_{XY}(\omega) = R_{XX}(\omega)H(\omega)$$

Given $R_{XX}(\tau) = e^{-2|\tau|}$

$$\therefore S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} e^{-2|\tau|} e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{0} e^{2\tau} e^{-i\omega\tau} d\tau + \int_{0}^{\infty} e^{-2\tau} e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{0} e^{(2-i\omega)\tau} d\tau + \int_{0}^{\infty} e^{-(2+i\omega)\tau} d\tau$$

$$= \left[\frac{e^{(2-i\omega)\tau}}{2-i\omega}\right]_{-\infty}^{0} + \left[\frac{e^{-(2+i\omega)\tau}}{-(2+i\omega)}\right]_{0}^{\infty}$$

$$= \frac{1}{2-i\omega}(1-0) - \frac{1}{2+i\omega}(0-1)$$

$$= \frac{1}{2-i\omega} + \frac{1}{2+i\omega} = \frac{4}{4+\omega^2}$$

$$\therefore H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt$$

$$= \int_{-\infty}^{\infty} 2e^{-t} e^{-i\omega t} dt$$

$$= 2\int_{0}^{\infty} e^{-(1+i\omega)t} dt = 2\left[\frac{e^{-(1+i\omega)t}}{-(1+i\omega)}\right]_{0}^{\infty}$$

$$= \frac{-2}{1+i\omega}[0-1] = \frac{2}{1+i\omega}$$

$$\therefore S_{XY}(\omega) = \frac{4}{4+\omega^2} \cdot \frac{2}{1+i\omega}$$

$$= \frac{8}{(2+i\omega)(2-i\omega)(1+i\omega)}$$

Let $\dfrac{8}{(2+i\omega)(2-i\omega)(1+i\omega)} = \dfrac{A}{2+i\omega} + \dfrac{B}{2-i\omega} + \dfrac{C}{1+i\omega}$

$\therefore 8 = A(2+i\omega)(1+i\omega) + B(2+i\omega)(1+i\omega) + C(2+i\omega)(2-i\omega)$

Put $\omega = i2$, then $8 = A(4)(-1) \Rightarrow A = -2$

$\omega = i$ then $8 = C(1)(3) \Rightarrow C = \dfrac{8}{3}$

$\omega = -i2$, then $8 = B(4)(3) \Rightarrow B = \dfrac{2}{3}$

$$\therefore S_{XY}(\omega) = \frac{-2}{2+i\omega} + \frac{\frac{2}{3}}{2-i\omega} + \frac{8/3}{1+i\omega}$$

Taking inverse Fourier transform

$$\therefore R_{XY}(\tau) = F^{-1}\left(\frac{-2}{2+i\omega}\right) + F^{-1}\left(\frac{2/3}{2-i\omega}\right) + F^{-1}\left(\frac{8/3}{1+i\omega}\right)$$

$$= -2.e^{-2\tau}\mu(\tau) + \frac{2}{3}e^{2\tau}\mu(-\tau) + \frac{8}{3}e^{-\tau}\mu(\tau)$$

Problem 22. If X(t) is a band limited process such that $S_{XX}(\omega) = 0$, where $|\omega| > \sigma$ prove that $2\left[R_{XX}(0) - R_{XX}(\tau)\right] \le \sigma^2\tau^2 R_{XX}(0)$

Solution:

Given $S_{XX}(\omega) = 0$, $|\omega| > \sigma$

$\Rightarrow S_{XX}(\omega) = 0$ if $\omega < -\sigma$ or $\omega > \sigma$

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{i\tau\omega}d\omega$$

$$= \frac{1}{2\pi}\int_{-\sigma}^{\sigma} S_{XX}(\omega)e^{i\tau\omega}d\omega$$

$$= \frac{1}{2\pi}\int_{-\sigma}^{\sigma} S_{XX}(\omega)(\cos\tau\omega + i\sin\tau\omega)\,dw$$

$$= \frac{1}{2\pi}\left\{\int_{-\sigma}^{\sigma} S_{XX}(\omega)\cos\tau\omega\,dw + i\int_{-\sigma}^{\sigma} S_{XX}(\omega)\sin\tau\omega\,dw\right\}$$

$$\therefore \int_{-\sigma}^{\sigma} S_{XX}(\omega)\cos\tau\omega\,d\omega = 2\int_{0}^{\sigma} S_{XX}(\omega)\cos\tau\omega\,d\omega \text{ and } \int_{-\sigma}^{\sigma} S_{XX}(\omega)\sin\tau\omega\,d\omega = 0$$

$$\therefore R_{XX}(\tau) = \frac{1}{2\pi}2\int_{0}^{\sigma} S_{XX}(\omega)\cos\tau\omega\,d\omega$$

$$= \frac{1}{\pi}\int_{0}^{\sigma} S_{XX}(\omega)\cos\tau\omega\,d\omega$$

$$\therefore R_{XX}(0) = \frac{1}{\pi}\int_{0}^{\sigma} S_{XX}(\omega)d\omega \qquad ...(1)$$

$$\therefore R_{XX}(0) - R_{XX}(\tau) = \frac{1}{\pi}\int_{-\sigma}^{\sigma} S_{XX}(\omega)\,d\omega - \frac{1}{\pi}\int_{-\sigma}^{\sigma} S_{XX}(\omega)\cos\omega\tau\,d\omega$$

$$= \frac{1}{\pi}\int_{-\sigma}^{\sigma} S_{XX}(\omega)(1-\cos\omega\tau)\,d\omega$$

$$= \frac{1}{\pi}\int_{-\sigma}^{\sigma} S_{XX}(\omega)\,2\sin^2\left(\frac{\omega\tau}{2}\right)d\omega$$

We know that $\sin^2\theta \le \theta^2$

$$\therefore 2\sin^2\left(\frac{\omega\tau}{2}\right) \le \left(\frac{\omega\tau}{2}\right)^2 = \frac{\omega^2\tau^2}{2} < \frac{\sigma^2\tau^2}{2} \quad \left[\because 0 \le \omega \le \sigma,\ \omega^2 \le \sigma^2\right]$$

$$\therefore R_{XX}(0) - R_{XX}(\tau) \le \frac{1}{\pi}\int_0^{\sigma} S_{XX}(\omega)\frac{\sigma^2\tau^2}{2}\,d\omega$$

$$\le \frac{1}{\pi}\frac{\sigma^2\tau^2}{2}\int_0^{\sigma} S_{XX}(\omega)\,d\omega$$

$$\le \frac{\tau^2\sigma^2}{2\pi}\int_0^{\sigma} S_{XX}(\omega)\,d\omega$$

$$\le \frac{\tau^2\sigma^2}{2\pi}\int_0^{\sigma} S_{XX}(\omega)\,d\omega$$

$$\le \frac{\sigma^2\tau^2}{2\pi}R_{XX}(0) \qquad \text{[Using (1)]}$$

$$\therefore 2\left[R_{XX}(0) - R_{XX}(\tau)\right] \le \sigma^2\tau^2 R_{XX}(0)$$

Problem 23. The autocorrelation function of the Poisson increment process is given by $R(\tau) = \begin{cases} \lambda^2 & \text{for } |\tau| > \in \\ \lambda^2 + \dfrac{\lambda}{\in}\left(1 - \dfrac{|\tau|}{\in}\right) & \text{for } |\tau| \le \in \end{cases}$. Prove that its spectral density is given by

$$S(\omega) = 2\pi\lambda^2\delta(\omega) + \frac{4\lambda\sin^2\frac{\omega t}{2}}{\;}.$$

Solution:

Given the autocorrelation function $R(\tau) = \begin{cases} \lambda^2 & \text{for } \tau > -\in \text{ or } \tau > \in \\ \lambda^2 + \dfrac{\lambda}{\in}\left(1 - \dfrac{|\tau|}{\in}\right) & \text{for } -\in \le \tau \le \in \end{cases}$

$\therefore$ The spectral density is given by $S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-i\tau\omega}d\tau$, by definition

$$= \int_{-\infty}^{-\in} R(\tau)e^{-i\tau\omega}d\tau + \int_{-\in}^{\in} R(\tau)e^{-i\tau\omega}d\tau + \int_{\in}^{\infty} R(\tau)e^{-i\tau\omega}d\tau$$

$$= \int_{-\infty}^{-\in} \lambda^2 e^{-i\tau\omega}d\tau + \int_{-\in}^{\in}\left[\lambda^2 + \frac{\lambda}{\in}\left(1 - \frac{|\tau|}{\in}\right)\right]e^{-i\tau\omega}d\tau + \int_{\omega}^{\infty}\lambda^2 e^{-i\tau\omega}d\tau$$

$$= \int_{-\infty}^{-\in} \lambda^2 e^{-i\tau\omega}d\tau + \int_{-\in}^{\in}\lambda^2 e^{-i\tau\omega}d\tau + \int_{\in}^{\infty}\lambda^2 e^{-i\tau\omega}d\tau + \int_{-\in}^{\in}\frac{\lambda}{\in}\left(1 - \frac{|\tau|}{\in}\right)e^{-i\tau\omega}d\tau$$

$$= \int_{-\infty}^{\infty} \lambda^2 e^{-i\tau\omega}d\tau + \frac{\lambda}{\in}\int_{-\in}^{\in}\left(1 - \frac{|\tau|}{\in}\right)e^{-i\tau\omega}d\tau$$

$$= F\left(\lambda^2\right) + \frac{\lambda}{\epsilon}\int_{-\epsilon}^{\epsilon}\left(1-\frac{|\tau|}{\epsilon}\right)\left(\cos\tau\omega - i\sin\tau\omega\right) d\tau$$

Where $F\left(\lambda^2\right)$ is the Fourier transform of λ^2

$$S(\omega) = F\left(\lambda^2\right) + \frac{\lambda}{\epsilon}\left\{\int_{-\epsilon}^{\epsilon}\left(1-\frac{|\tau|}{\epsilon}\right)\cos\tau\omega\, d\tau - i\int_{-\epsilon}^{\epsilon}\left(1-\frac{|\tau|}{\epsilon}\right)\sin\tau\omega\, d\tau\right\} \quad .....(1)$$

But $\left(1-\frac{|\tau|}{\epsilon}\right)\cos\tau\omega$ is an even function of τ and $\left(1-\frac{|\tau|}{\epsilon}\right)\sin\tau\omega$ is an odd function of τ

$$\therefore \int_{-\epsilon}^{\epsilon}\left(1-\frac{|\tau|}{\epsilon}\right)\sin\tau\omega\, d\tau = 0 \text{ and } \int_{-\epsilon}^{\epsilon}\left(1-\frac{|\tau|}{\epsilon}\right)\cos\tau\omega\, d\tau = 2\int_0^{\epsilon}\left(1-\frac{\tau}{\epsilon}\right)\cos\tau\omega\, d\tau \quad \left(\because \tau > 0; |\tau| = \tau\right)$$

$$S(\omega) = F\left(\lambda^2\right) + \frac{\lambda}{\epsilon}2\int_0^{\epsilon}\left(1-\frac{\tau}{\epsilon}\right)\cos\tau\omega\, d\tau$$

$$= F\left(\lambda^2\right) + \frac{2\lambda}{\epsilon}\left[\left(1-\frac{\tau}{\epsilon}\right)\left(\frac{\sin\tau\omega}{\omega}\right) - \left(\frac{-1}{\epsilon}\right)\left(\frac{-\cos\tau\omega}{\omega^2}\right)\right]_0^{\epsilon} \quad \text{[by Bernoulli's formula]}$$

$$= F\left(\lambda^2\right) + \frac{2\lambda}{\epsilon}\left[\frac{1}{\omega}\left(1-\frac{\tau}{\epsilon}\right)\sin\tau\omega - \frac{1}{\epsilon\omega^2}\cos\tau\omega\right]_0^{\epsilon}$$

$$= F\left(\lambda^2\right) + \frac{2\lambda}{\epsilon}\left[0 - \frac{1}{\epsilon\omega^2}\left(\cos\epsilon\omega - \cos 0\right)\right]$$

$$= F\left(\lambda^2\right) + \frac{2\lambda}{\epsilon}\frac{1}{\epsilon\omega^2}\left[1-\cos\epsilon\omega\right]$$

$$= F\left(\lambda^2\right) + \frac{2\lambda}{\epsilon^2\omega^2}.2\sin^2\frac{\epsilon\omega}{2}$$

$$S(\omega) = F\left(\lambda^2\right) + \frac{4\lambda}{\epsilon^2\omega^2}\sin^2\frac{\epsilon\omega}{2} \quad ....(1)$$

To find the value of $F\left(\lambda^2\right)$, we shall find the inverse Fourier transform of $S(\omega)$,

$$R(\tau) = F^{-1}\left(S(\omega)\right)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{i\omega\tau}d\omega$$

Consider $S(\omega) = 2\pi\lambda^2\delta(\omega)$, where $\delta(\omega)$ is the unit impulse function.

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\lambda^2\delta(\omega)e^{i\omega\tau}d\omega$$

$$= \lambda^2\int_{-\infty}^{\infty}\delta(\omega)e^{i\omega\tau}d\omega$$

$$= \lambda^2.1 \qquad \left[\because \int_{-\infty}^{\infty}\varphi(t)\delta(t)\,dt = \varphi(0)\right]$$

$= \lambda^2 \qquad \Rightarrow \int_{-\infty}^{\infty} e^{i\omega\tau}\delta(\omega)\,d\omega = e^0 = 1 \;\; as \; \varphi(\omega) = e^{i\tau\omega}]$

Thus $\lambda^2 = R(\tau)$ Taking Fourier transform

$F(\lambda^2) = F(R(\tau)) = S(\omega) = 2\pi\lambda^2\delta(\omega)$

Substituting in (1) we get $S(\omega) = 2\pi\lambda^2\delta(\omega) + \frac{4\lambda}{\epsilon^2\omega^2}\sin^2\left(\frac{\omega\epsilon}{2}\right)$.

Problem 24. Suppose X(t) be the input process to a linear system with autocorrelation $R_{XX}(\tau) = 3\delta(\tau)$ and the impulse response $H(\omega) = \frac{1}{6 + i\omega}$, then find(i) the autocorrelation of the output process Y(t). (ii) the power spectral density of Y(t).

Solution:

Given $R_{XX}(\tau) = 3\delta(\tau)$ and $H(\omega) = \frac{1}{6 + i\omega}$

$\therefore S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-i\omega\tau}d\tau$

$= \int_{-\infty}^{\infty} 3\delta(\tau)e^{-i\omega\tau}d\tau$

$= 3\int_{-\infty}^{\infty} \delta(\tau)e^{-i\omega\tau}d\tau$

We know $\int_{-\infty}^{\infty} \delta(\tau)\varphi(\tau) = \varphi(0)$

Here $\varphi(\tau) = e^{-i\omega\tau} \;\; \therefore \varphi(0) = 1$

$\therefore S_{XX}(\omega) = 3.1 = 3$

We know the spectral relation between input and output process is $S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$

But $|H(\omega)|^2 = \frac{1}{36 + \omega^2}$

$\therefore S_{YY}(\omega) = \frac{3}{36 + \omega^2}$ which is the power spectral density of $Y(t)$

Now the autocorrelation of $Y(t)$ is $R_{YY}(\tau) = F^{-1}(S_{YY}(\omega))$

$R_{YY}(\tau) = F^{-1}\left(\frac{3}{36 + \omega^2}\right)$

We know $F^{-1}\left(\frac{2\alpha}{\alpha^2 + \omega^2}\right) = e^{-\alpha|\tau|}$

$\therefore R_{YY}(\tau) = \frac{3}{2.6}F^{-1}\left(\frac{2.6}{6^2 + \omega^2}\right) \qquad [Here\; \alpha = 6]$

$$= \frac{1}{4} e^{-6|\tau|}$$

(ii) $S_{YY}(\omega) = \int_{-\infty}^{\infty} R_{YY}(\tau) e^{-i\omega\tau} d\tau$

$$= \int_{-\infty}^{\infty} \frac{1}{4} e^{-6|\tau|} e^{-i\tau\omega} d\tau$$

$$= \frac{1}{4}\left\{ \int_{-\infty}^{0} e^{(6 - i\omega)\tau} d\tau + \int_{0}^{\infty} e^{-(6+i\omega)\tau} d\tau \right\}$$

$$= \frac{1}{4}\left\{ \left[\frac{e^{(6-i\omega)\tau}}{6 - i\omega} \right]_{-\infty}^{0} + \left[\frac{e^{-(6+i\omega)\tau}}{-(6 + i\omega)} \right]_{0}^{\infty} \right\}$$

$$= \frac{1}{4}\left\{ \frac{1}{6 - i\omega}(1 - 0) - \frac{1}{6 + i\omega}(0 - 1) \right\}$$

$$= \frac{1}{4}\left\{ \frac{1}{6 - i\omega} + \frac{1}{6 + i\omega} \right\}$$

$$= \frac{1}{4}\left\{ \frac{(6 + i\omega) + (6 - i\omega)}{(6 - i\omega)(6 + i\omega)} \right\}$$

$$= \frac{1}{4}\left\{ \frac{12}{36 + \omega^2} \right\} = \frac{3}{36 + \omega^2}.$$

Problem 25. Show that the power spectrum $S_{YY}(\omega)$ of the output of a linear system with system function $H(\omega)$ is given by $S_{YY}(\omega) = S_{XX}(\omega)|H(\omega)|^2$ where $S_{XX}(\omega)$ is the power spectrum of the input.

Solution:

If $\{X(t)\}$ is a WSS and if $y(t) = \int_{-\infty}^{\infty} h(\alpha) X(t - \alpha) d\alpha$

We shall prove that $S_{YY}(\omega) = S_{XY}(\omega)|H(\omega)|^2$.

Consider $Y(t) = \int_{-\infty}^{\infty} X(t - \alpha) h(\alpha) d\alpha$

$$X(t + \tau) Y(t) = \int_{-\infty}^{\infty} X(t + \tau) X(t - \alpha) h(\alpha) d\alpha$$

$$E\left[X(t + \tau) Y(t) \right] = \int_{-\infty}^{\infty} E\left[X(t + \tau) X(t - \alpha) \right] h(\alpha) d\alpha$$

$$R_{YX}(-\tau) = \int_{-\infty}^{\infty} R_{XX}(\tau + \alpha) h(\alpha) d\alpha$$

$$R_{XY}(\tau) = \int_{-\infty}^{\infty} R_{XX}(\tau - \beta) h(-\beta) d\beta$$

$R_{XY}(\tau) = R_{XX}(\tau) * h(-\tau)$

$Y(t)Y(t-\tau) = \int_{-\infty}^{\infty} X(t-\alpha)Y(t-\tau)h(\alpha)\,d\alpha$

$\therefore E\left[Y(t)Y(t-\tau)\right] = \int_{-\infty}^{\infty} R_{XY}(\tau-\alpha)h(\alpha)\,d\alpha$

Assuming that $\{X(t)\}$ & $\{Y(t)\}$ are jointly WSS

$R_{YY}(\tau) = R_{XY}(\tau) * h(\tau)$ ------------ (2)

Taking Fourier transform of (1) we get

$S_{XY}(\omega) = S_{XX}(\omega)\, H^*(\omega)$ --------(3)

Where $H^*(\omega)$ is the conjugate of $H(\omega)$

Taking Fourier transform of (2) we get

$S_{YY}(\omega) = S_{XY}(\omega)\, H(\omega)$ ------------(4)

Inserting (3) in (4)

$S_{YY}(\omega) = S_{XX}(\omega)\, H^*(\omega) H(\omega)$

$S_{YY}(\omega) = S_{XX}(\omega)\, |H(\omega)|^2$

Problem 26. A system has an impulse response $h(t) = e^{-\beta t}U(t)$, find the power spectral density of the output $Y(t)$ corresponding to the input $X(t)$.

Solution:

Given $h(t) = e^{-\beta t}, t \geq 0$

$H(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-i\omega t} dt$

$H(\omega) = \int_{0}^{\infty} e^{-\beta t} e^{-i\omega t} dt$

$= \int_{0}^{\infty} e^{-t(\beta + i\omega)} dt$

$= \left[\dfrac{e^{-t(\beta+i\omega)}}{-(\beta + i\omega)}\right]_0^{\infty}$

$H(\omega) = \dfrac{1}{\beta + i\omega}$

$H^*(\omega) = \dfrac{1}{\beta - i\omega}$

$|H(\omega)|^2 = H(\omega) H^*(\omega)$

$= \left(\dfrac{1}{\beta + i\omega}\right)\left(\dfrac{1}{\beta - i\omega}\right)$

$$= \frac{1}{\beta^2 + \alpha^2}$$

$$\therefore S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$

$$S_{YY}(\omega) = \frac{S_{XX}(\omega)}{\beta^2 + \alpha^2}$$

Problem 27. If $X(t)$ is the input voltage to a circuit and $Y(t)$ is the output voltage, $\{X(t)\}$ is a stationary random process with $\mu_x = 0$ and $R_{XX}(\tau) = e^{-2|\tau|}$. Find μ_y, $S_{XX}(\omega)$ and $S_{YY}(\omega)$, if the system function is given by $H(\omega) = \frac{1}{\omega^2 + 2^2}$.

Solution:

Given Mean $[X(t)] = \mu_X = 0$

$$Y(t) = \int_{-\infty}^{\infty} h(\alpha) X(t-\alpha)\, d\alpha$$

$$E[Y(t)] = \int_{-\infty}^{\infty} h(\alpha) E[X(t-\alpha)]\, d\alpha = 0$$

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} e^{-2|\tau|} e^{-i\omega\tau} d\tau = \int_{-\infty}^{\infty} e^{-2|\tau|} \cos\omega\tau\, d\tau = 2\int_{0}^{\infty} e^{-2\tau} \cos\omega\tau\, d\tau$$

$$S_{XX}(\omega) = 2\left[\frac{2}{\omega^2 + 4}\right] = \frac{4}{\omega^2 + 4}$$

$$H(\omega) = \frac{1}{\omega^2 + 4}$$

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega) = \left(\frac{1}{\omega^2 + 4}\right)^2 \frac{4}{\omega^2 + 4}$$

$$S_{YY}(\omega) = \frac{4}{(\omega^2 + 4)^2}$$

Problem 28. $X(t)$ is the input voltage to a circuit (system) and $Y(t)$ is the output voltage. $\{X(t)\}$ is a stationary random process with $\mu_x = 0$ and $R_{XX}(\tau) = e^{-\alpha|\tau|}$. Find μ_y, $S_{yy}(\omega)$ & $R_{yy}(\tau)$ if the power transfer function is $H(\omega) = \frac{R}{R + iL\omega}$

Solution:

$$Y(t) = \int_{-\infty}^{\infty} h(\alpha) X(t-\alpha)\, d\alpha$$

$$E\left[Y(t)\right] = \int_{-\infty}^{\infty} h(\alpha)\, E\left[X(t-\alpha)\, d\alpha\right]$$

$$E\left[Y(t)\right] = 0$$

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{0} e^{\alpha\tau} e^{-i\omega\tau}\, d\tau + \int_{0}^{\infty} e^{-\alpha\tau} e^{-i\omega\tau}\, d\tau$$

$$= \left[\frac{e^{(\alpha - i\omega)\tau}}{\alpha - i\omega}\right]_{-\infty}^{0} + \left[\frac{e^{-(\alpha + i\omega)\tau}}{-(\alpha + i\omega)}\right]_{0}^{\infty} = \frac{1}{\alpha - i\omega} + \frac{1}{\alpha + i\omega} = \frac{2\alpha}{\alpha^2 + \omega^2}$$

$$S_{YY}(\omega) = S_{XX}(\omega)\, \left|H(\omega)\right|^2$$

$$= \frac{2\alpha}{\alpha^2 + \omega^2}\, \frac{R}{R^2 + L^2\omega^2}$$

Consider,

$$\frac{2\alpha R^2}{\left(\alpha^2 + \omega^2\right)\left(R^2 + L^2\omega^2\right)} = \frac{A}{\alpha^2 + \omega^2} + \frac{B}{R^2 + L^2\omega^2}$$

By partial fractions

$$S_{YY}(\omega) = \frac{2\alpha\left(\frac{R^2}{R^2 - L^2\alpha^2}\right)}{\alpha^2 + \omega^2} + \frac{2\alpha\left(\frac{R^2}{\alpha^2 - \frac{R^2}{L^2}}\right)}{R^2 + L^2\omega^2}$$

$$= \frac{2\alpha\left(\frac{R}{L}\right)^2}{\left(\frac{R}{L}\right)^2 - \alpha^2} \cdot \frac{1}{\alpha^2 + \omega^2} + \frac{2\alpha\left(\frac{R^2}{L^2}\right)}{\alpha^2 - \left(\frac{R}{L}\right)^2} \cdot \frac{1}{\left(\frac{R}{L}\right)^2 + \omega^2}$$

$$= \lambda \cdot \frac{1}{\alpha^2 + \omega^2} + \mu \cdot \frac{1}{\left(\frac{R}{L}\right)^2 + \omega^2}$$

$$R_{YY}(\tau) = \frac{\lambda}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega\tau}}{\alpha^2 + \omega^2}\, d\omega + \frac{\mu}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega\tau}}{\left(\frac{R}{L}\right)^2 + \omega^2}\, d\omega$$

By contour integration technique we know that

$$\int_{-\infty}^{\infty} \frac{e^{iaz}}{z^2 + b^2}\, dz = \frac{\pi}{b}\, e^{-ab},\ a > 0$$

$$\therefore R_{YY}(\tau) = \frac{\left(\frac{R}{L}\right)^2}{\left(\frac{R}{L}\right)^2 - \alpha^2} e^{-\alpha|\tau|} + \frac{\left(\frac{R}{L}\right)^2 \alpha}{\alpha^2 - \left(\frac{R}{L}\right)^2} e^{\left(-\frac{R}{L}|\tau|\right)}.$$

Problem 29. $X(t)$ is the i/P voltage to a circuit and $Y(t)$ is the O/P voltage. $X(t)$ is a stationary random process with zero mean and autocorrelation $R_{XX}(\tau) = e^{-2|\tau|}$. Find the mean of $Y(t)$ and its PSD if the system function $H(\omega) = \frac{1}{J\omega + 2}$.

Solution:

$$H(\omega) = \frac{1}{J\omega + 2}$$

$$\Rightarrow H(0) = \frac{1}{2}$$

$$E[Y(t)] = E[X(t)].H(0) = 0$$

$$|H(\omega)|^2 = \frac{1}{\omega^2 + 4}$$

$$S_{XX}(\omega) = F[R_{XX}(\tau)] = F\left[e^{-2|\tau|}\right] = \frac{4}{\omega^2 + 4}$$

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega) = \frac{4}{\left(\omega^2 + 4\right)^2}$$

$$\therefore E\left(Y^2\right) = \int_{-2}^{0}(2+\tau)\left(9 + 2e^{\tau}\right)d\tau + \int_{0}^{2}(2-\tau)\left(9 + 2e^{-\tau}\right)d\tau$$

$$= \left[18\tau + 4e^{\tau} + \frac{9\tau^2}{2} + 2e^{\tau}(\tau - 1)\right]_{-2}^{0} + \left[18\tau - 4e^{\tau} - \frac{9\tau^2}{2} 2e^{-\tau}(\tau + 1)\right]_{0}^{2}$$

$$\therefore E\left(Y_2\right) = 40.542$$

$$Var(Y) = E\left(Y^2\right) - [E(Y)]^2$$

$$= 40.542 - 36 = 4.542$$

Problem 30. Consider a system with transfer function $\frac{1}{1 + i\omega}$. An input signal with autocorrelation function $m\delta(\tau) + m^2$ is fed as input to the system. Find the mean and mean-square value of the output.

Solution:

Given, $H(\omega) = \frac{1}{1 + i\omega}$ and $R_{XX}(\tau) = m\delta(\tau) + m^2$

$$S_{XX}(\omega) = m + 2\pi m^2 \delta(\omega)$$

We know that, $S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$

$$= \left|\frac{1}{1+i\omega}\right|^2 \left[m + 2\pi m^2 \delta(\omega)\right]$$

$$= \frac{1}{1+\omega^2}\left[m + 2\pi m^2 \delta(\omega)\right]$$

$R_{YY}(\tau)$ is the Fourier inverse transform of $S_{YY}(\omega)$.

So, $R_{YY}(\tau) = \frac{m}{2} e^{-|\tau|} + m^2$

We know that $\lim_{\tau \to \infty} R_{XX}(\tau) = \overline{X}^2$

So $\overline{X}^2 = M^2$

$\overline{X} = m$

Also $H(0) = 1$

We know that $\overline{Y} = 1, m = m$

Mean-square value of the output $= \overline{Y^2} = R_{YY}(0) = \frac{m}{2} + m^2$

Problem 31. If the input to a time-invariant, stable, linear system is a WSS process, prove that the output will also be a WSS process.

Solution:

Let $X(t)$ be a WSS process for a linear time variant stable system with $Y(t)$ as the output process.

Then $Y(t) = \int_{-\infty}^{\infty} h(u) X(t-u)\, du$ where h(t) is weighting function or unit impulse response.

$$\therefore E[Y(t)] = \int_{-\infty}^{\infty} E[h(u) X(t-u)]\, du$$

$$= \int_{-\infty}^{\infty} h(u) E[X(t-u)]\, du$$

Since $X(t)$ is a WSS process, $E[X(t)]$ is a constant μ_X for any t.

$$\therefore E[X(t-u)] = \mu_X$$

$$\therefore E[Y(t)] = \int_{-\infty}^{\infty} h(u) \mu_X\, du$$

$$= \mu_X \int_{-\infty}^{\infty} h(u)\, du$$

Since the system is stable, $\int_{-\infty}^{\infty} h(u)\, du$ is finite

$\therefore E[Y(t)]$ is a constant.

Now $R_{YY}(t,t+\tau) = E[Y(t)Y(t+\tau)]$

$$= E\left[\int_{-\infty}^{\infty} h(u_1) X(t-u_1)\, du_1 \int_{-\infty}^{\infty} h(u_2) X(t+\tau-u_2)\, du_2\right]$$

$$= E\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u_1) h(u_2) X(t-u_1) X(t+\tau-u_2)\, du_1 u_2\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u_1) h(u_2) E[X(t-u_1) X(t+\tau-u_2)]\, du_1 u_2$$

Since $X(t)$ is a WSS process, auto correlation function is only a function time difference.

$$\therefore R_{YY}(t,t+\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u_1) h(u_2) R_{XX}(\tau + u_1 - u_2)\, du_1\, du_2$$

When this double integral is evaluated by integrating w.r.t. to u_1, u_2, the R.H.S is only a function of τ .

$\therefore R_{YY}(t,t+\tau)$ is only a function of time difference τ .

Hence $Y(t)$ is a WSS process.

Problem 32. Let $X(t)$ be a WSS and if $Y(t) = \int_{-\infty}^{\infty} h(u) X(t-u)\, du$ then show that

a) $R_{XY}(\tau) = h(\tau) * R_{XX}(\tau)$

b) $R_{YX}(\tau) = h(-\tau) * R_{XX}(\tau)$

c) $R_{yy}(\tau) = h(\tau) * R_{xy}(\tau)$

Where * denotes the convolution and $H^*(\omega)$ is the complex conjugate of $H(\omega)$.

Solution:

Given $X(t)$ is WSS $\therefore E[X(t)]$ is constant and

$R_{XX}(t,t+\tau) = R_{XX}(\tau)$

$$Y(t) = \int_{-\infty}^{\infty} h(u) X(t-u)\, du$$

Now $R_{XY}(t,t+\tau) = E[X(t)Y(t+\tau)]$

$$= E\left[X(t) \int_{-\infty}^{\infty} h(u).X(t+\tau-u)\, du\right]$$

$$= E\left[\int_{-\infty}^{\infty} h(u) X(t) X(t+\tau-u)\, du\right] = \int_{-\infty}^{\infty} h(u).E[X(t) X(t+\tau-u)]\, du$$

Since $X(t)$ is a WSS Process,

$E[X(t) X(t+\tau-u)] = R_{XX}(\tau-u)$

$$\therefore R_{XY}(t,t+\tau) = \int_{-\infty}^{\infty} h(u) R_{XX}(\tau-u)\, du$$

$\Rightarrow R_{XY}(\tau) = R_{XX}(\tau) * h(\tau)$

(b). Now $R_{YX}(\tau) = R_{XY}(-\tau)$

$= R_{XY}(-\tau) * h(-\tau)$ $\qquad [from\ (i)]$

$= R_{XX}(\tau) * h(-\tau)$ $\qquad$ [Since $R_{XX}(\tau)$ is an even function of τ]

(c). $R_{YY}(t, t-\tau) = E\left[Y(t)Y(t-\tau)\right]$

$$= E\left[\int_{-\infty}^{\infty} h(u)\, X(t-u)\, du\, Y(t-\tau)\right]$$

$$= E\left[\int_{-\infty}^{\infty} X(t-u) Y(t-\tau) h(u)\, du\right]$$

$$= \int_{-\infty}^{\infty} E\left[X(t-u)Y(t-\tau)\right] h(u)\, du = \int_{-\infty}^{\infty} R_{XY}(\tau-u) h(u)\, du$$

It is a function of τ only and it is true for any τ.

$\therefore R_{YY}(\tau) = R_{XY}(\tau) * h(\tau)$

Problem 33. Prove that the mean of the output of a linear system is given by $\mu_Y = H(0)\mu_X$, where $X(t)$ is WSS.

Solution:

We know that the input $X(t)$, output $Y(t)$ relationship of a linear system can expressed as a convolution $Y(t) = h(t) * X(t)$

$$= \int_{-\infty}^{\infty} h(u)(t-u)\, du$$

Where $h(t)$ is the unit impulse response of the system.

$\therefore$the mean of the output is

$$E\left[Y(t)\right] = E\left[\int_{-\infty}^{\infty} h(u)\, X(t-u)\, du\right] = \int_{-\infty}^{\infty} h(u) E\left[X(t-u)\right] du$$

Since X(t) is WSS, $E\left[X(t)\right] = \mu_X$ is a constant for any t.

$E\left[X(t-u)\right] = \mu_X$

$$\therefore E\left[Y(t)\right] = \int_{-\infty}^{\infty} h(u)\mu_X\, du = \mu_X \int_{-\infty}^{\infty} h(u)\, du$$

We know $H(\omega)$ is the Fourier transform of $h(t)$.

i.e. $H(\omega) = \int_{-\infty}^{\infty} h(t)\, dt$

put $\omega = 0$ $\therefore H(0) = \int_{-\infty}^{\infty} h(t)\, dt = \int_{-\infty}^{\infty} h(u)\, du$

$\therefore E\left[Y(t)\right] = \mu_X H(0)$.

UNIT –I RANDOM VARIABLES

PART-A

Problem1. X and Y are independent random variables with variance 2 and 3. Find the variance of $3X+4Y$.

Solution:

$V(3X+4Y) = 9Var(X)+16Var(Y)+24Cov(XY)$

$= 9\times 2+16\times 3+0 \quad (\because X \,\&\, Y \textit{ are independent } \operatorname{cov}(XY)=0)$

$= 18+48 = 66.$

Problem 2. A Continuous random variable X has a probability density function $F(x) = 3x^2;$ $0 \le x \le 1$. Find 'a' such that $P(x \le a) = P(x > a)$

Solution:

We know that the total probability =1

Given $P(X \le a) = P(X > a) = K(say)$

Then $K+K=1$

$$K = \frac{1}{2}$$

ie $P(X \le a) = \frac{1}{2} \,\&\, P(X > a) = \frac{1}{2}$

Consider $P(X \le a) = \frac{1}{2}$

$$\text{i.e. } \int_0^a f(x)\,dx = \frac{1}{2}$$

$$\int_0^a 3x^2 dx = \frac{1}{2}$$

$$3\left(\frac{x^3}{3}\right)_0^a = \frac{1}{2}$$

$$a^3 = \frac{1}{2}$$

$$a = \left(\frac{1}{2}\right)^{1/3}.$$

Problem 3. A random variable X has the p.d.f $f(x)$ given by $f(x) = \begin{cases} Cxe^{-x}; & if\ x > 0 \\ 0 & ; if\ x \le 0 \end{cases}$

Find the value of C and cumulative density function of X.

Solution:

Since $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^{\infty} Cxe^{-x}dx = 1$$

$$C\left[x\left(-e^{-x}\right)-\left(e^{-x}\right)\right]_0^{\infty} = 1$$

$$C = 1$$

$$\therefore f(x) = \begin{cases} xe^{-x}; & x > 0 \\ 0 \quad; & x \le 0 \end{cases}$$

$$C.D.F\ F(x) = \int_0^x f(x)dt = \int_0^x te^{-t}dt = \left[-te^{-t} - e^{-t}\right]_0^x = -xe^{-x} - e^{-x} + 1$$

$$= 1-(1+x)e^{-x}.$$

Problem 4. If a random variable X has the p.d.f $f(x) = \begin{cases} \frac{1}{2}(x+1); & -1 < x < 1 \\ 0 \quad; & otherwise \end{cases}$.

Find the mean and variance of X.

Solution:

$$\text{Mean} = \int_{-1}^{1} xf(x)\,dx = \frac{1}{2}\int_{-1}^{1} x(x+1)dx = \frac{1}{2}\int_{-1}^{1}\left(x^2 + x\right)dx$$

$$= \frac{1}{2}\left(\frac{x^3}{3} + \frac{x^2}{2}\right)_{-1}^{1} = \frac{1}{3}$$

$$\mu_2' = \int_{-1}^{1} x^2 f(x)dx = \frac{1}{2}\int_{-1}^{1}\left(x^3 + x^2\right)dx = \frac{1}{2}\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\frac{1}{4} + \frac{1}{3} - \frac{1}{4} + \frac{1}{3}\right]$$

$$= \frac{1}{2}\cdot\frac{2}{3} = \frac{1}{3}$$

$$Variance = \mu_2' - \left(\mu_1'\right)^2$$

$$= \frac{1}{3} - \frac{1}{9} = \frac{3-1}{9} = \frac{2}{9}.$$

Problem 5. A random variable X has density function given by $f(x) = \begin{cases} 2e^{-2x}; & x \ge 0 \\ 0 \quad; & x < 0 \end{cases}$.

Find m.g.f

Solution:

$$M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx} f(x)\,dx = \int_0^\infty e^{tx}\, 2e^{-2x} dx$$

$$= 2\int_0^\infty e^{(t-2)x} dx$$

$$= 2\left[\frac{e^{(t-2)x}}{t-2}\right]_0^\infty = \frac{2}{2-t}, t < 2.$$

Problem 6. Criticise the following statement: "The mean of a Poisson distribution is 5 while the standard deviation is 4".

Solution: For a Poisson distribution mean and variance are same. Hence this statement is not true.

Problem 7. Comment the following: "The mean of a binomial distribution is 3 and variance is 4

Solution:

In binomial distribution, mean > variance but Variance < Mean

Since Variance = 4 & Mean = 3, the given statement is wrong.

Problem8. If X and Y are independent binomial variates $B\left(5, \frac{1}{2}\right)$ and $B\left(7, \frac{1}{2}\right)$ find $P[X + Y = 3]$

Solution:

$X + Y$ is also a binomial variate with parameters $n_1 + n_2 = 12$ & $p = \frac{1}{2}$

$$\therefore P[X + Y = 3] = 12C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^9$$

$$= \frac{55}{2^{10}}$$

Problem 9. If X is uniformly distributed with Mean 1 and Variance $\frac{4}{3}$, find $P[X > 0]$

Solution:

If X is uniformly distributed over (a,b), then

$$E(X) = \frac{b+a}{2} \text{ and } V(X) = \frac{(b-a)^2}{12}$$

$$\therefore \frac{b+a}{2} = 1 \Rightarrow a + b = 2$$

$$\Rightarrow \frac{(b-a)^2}{12} = \frac{4}{3} \Rightarrow (b-a)^2 = 16$$

$\Rightarrow a + b = 2$ & $b - a = 4$ We get $b = 3, a = -1$

$\therefore a = -1$ & $b = 3$ and probability density function of x is

$$f(x) = \begin{cases} \frac{1}{4}; & -1 < x < 3 \\ 0; & Otherwise \end{cases}$$

$$P[x < 0] = \int_{-1}^{0} \frac{1}{4} dx = \frac{1}{4}[x]_{-1}^{0} = \frac{1}{4}.$$

Problem 10. State the memoryless property of geometric distribution.

Solution:

If X has a geometric distribution, then for any two positive integer 'm' and 'n'

$P\left[X > m+n / X > m\right] = P[X > n]$.

Problem 11. X is a normal variate with $mean = 30$ and $S.D = 5$

Find the following $P[26 \le X \le 40]$

Solution:

$X \sim N(30, 5^2)$

$\therefore \mu = 30 \ \& \ \sigma = 5$

Let $Z = \frac{X - \mu}{\sigma}$ be the standard normal variate

$$P[26 \le X \le 40] = P\left[\frac{26-30}{5} \le Z \le \frac{40-30}{5}\right]$$

$$= P[-0.8 \le Z \le 2] = P[-0.8 \le Z \le 0] + P[0 \le Z \le 2]$$

$$= P[0 \le Z \ 0.8] + [0 \le z \le 2]$$

$$= 0.2881 + 0.4772 = 0.7653.$$

Problem 12. If X is a $N(2,3)$ Find $P\left[Y \ge \frac{3}{2}\right]$ where $Y + 1 = X$.

Solution:

$$P\left[Y \ge \frac{3}{2}\right] = P\left[X - 1 \ge \frac{3}{2}\right]$$

$$= P[X \ge 2.5] = P[Z \ge 0.17]$$

$$= 0.5 - P[0 \le Z \le 0.17]$$

$$= 0.5 - 0.0675 = 0.4325$$

Problem 13. If the probability is $\frac{1}{4}$ that a man will hit a target what is the chance that he will hit the target for the first time in the 7^{th} trial?

Solution:

The required probability is

$$P[FFFFFFS] = P(F)P(F)P(F)P(F)P(F)P(F)P(S)$$

$$= q^6 p = \left(\frac{3}{4}\right)^6 \cdot \left(\frac{1}{4}\right) = 0.0445.$$

Hence p = Probability of hitting target and $q = 1 - p$.

Problem 14. A random variable X has an exponential distribution defined by p.d.f. $f(x) = e^{-x},\ 0 < x < \infty$. Find the density function of $Y = 3X + 5$.

Solution:

$$y = 3x + 5 \Rightarrow \frac{dy}{dx} = 3 \Rightarrow \frac{dx}{dy} = \frac{1}{3}$$

P.d.f of y $h_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$

$$h_Y(y) = \frac{1}{3}e^{-x}.$$

Using $x = \frac{y-5}{3}$ we get $h_Y(y) = \frac{1}{3}e^{-\left(\frac{y-5}{3}\right)},\ y > 5\ (\because x > 0 \Rightarrow y > 5)$

Problem 15. If X is a normal variable with zero mean and variance σ^2, Find the p.d.f of $y = e^{-x}$

Solution:

Given $Y = e^{-x}$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}$$

$$h_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2\sigma^2}(\log y)^2} \times \frac{1}{y}.$$

PART-B

Problem 16. A random variable X has the following probability function:

Values of X,

X	:	0	1	2	3	4	5	6	7
$P(X)$	:	0	K	$2K$	$2K$	$3K$	K^2	$2K^2$	$7K^2 + K$

Find (i) K, (ii) Evaluate $P(X < 6), P(X \geq 6)$ and $P(0 < X < 5)$

(iii). Determine the distribution function of X.

(iv). $P(1.5 < X < 4.5 / X > 2)$

(v). $E(3x - 4)$, $Var(3x - 4)$

Solution(i):

Since $\sum_{x=0}^{7} P(X) = 1,$

$$K + 2K + 2K + 3K + K^2 + 2K^2 + 7K^2 + K = 1$$

$$10K^2 + 9K - 1 = 0$$

$$K = \frac{1}{10} \quad or \quad K = -1$$

As $P(X)$ cannot be negative $K = \frac{1}{10}$

Solution(ii):

$$P(X<6)=P(X=0)+P(X=1)+\dots+P(X=5)$$
$$=\frac{1}{10}+\frac{2}{10}+\frac{2}{10}+\frac{3}{10}+\frac{1}{100}+\dots=\frac{81}{100}$$

Now $P(X\ge 6)=1-P(X<6)$

$$=1-\frac{81}{100}=\frac{19}{100}$$

Now $P(0<X<5)=P(X=1)+P(X=2)+P(X=3)=P(X=4)$

$$=K+2K+2K+3K$$
$$=8K=\frac{8}{10}=\frac{4}{5}.$$

Solution(iii):

The distribution of X is given by $F_X(x)$ defined by

$$F_X(x)=P(X\le x)$$

X	:	0	1	2	3	4	5	6	7
$F_X(x)$	:	0	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{5}{10}$	$\frac{4}{5}$	$\frac{81}{100}$	$\frac{83}{100}$	1

Problem 17. (a) If $P(x)=\begin{cases}\frac{x}{15} & ;x=1,2,3,4,5\\ 0 & ;elsewhere\end{cases}$

Find (i) $P\{X=1 or 2\}$ and (ii) $P\{1/2<X<5/2/x>1\}$

(b) X is a continuous random variable with pdf given by

$$F(X)=\begin{cases}Kx & in\ \ 0\le x\le 2\\ 2K & in\ \ 2\le x\le 4\\ 6K-Kx & in\ \ 4\le x\le 6\\ 0 & elsewhere\end{cases}$$

Find the value of K and also the cdf $F_X(x)$.

Solution:

(a) i) $P(X=1\ or\ 2)=P(X=1)+P(X=2)$

$$=\frac{1}{15}+\frac{2}{15}=\frac{3}{15}=\frac{1}{5}$$

ii) $$P\left(\frac{1}{2}<X<\frac{5}{2}/x>1\right)=\frac{P\left\{\left(\frac{1}{2}<X<\frac{5}{2}\right)\cap(X>1)\right\}}{P(X>1)}$$

$$=\frac{P\{(X=1 or 2)\cap(X>1)\}}{P(X>1)}$$

$$= \frac{P(X=2)}{1-P(X=1)}$$

$$= \frac{2/15}{1-(1/15)} = \frac{2/15}{14/15} = \frac{2}{14} = \frac{1}{7}.$$

Since $\int_{-\infty}^{\infty} F(x)\,dx = 1$

$$\int_0^2 Kx\,dx + \int_2^4 2K\,dx + \int_4^6 (6k - kx)\,dx = 1$$

$$K\left[\left(\frac{x^2}{2}\right)_0^2 + (2x)_2^4 + \left(6x - \frac{x^2}{2}\right)_4^6\right] = 1$$

$$K\left[2 + 8 - 4 + 36 - 18 - 24 + 8\right] = 1$$

$$8K = 1 \quad K = \frac{1}{8}$$

We know that $F_X(x) = \int_{-\infty}^{x} f(x)\,dx$

If $x < 0$, then $F_X(x) = \int_{-\infty}^{x} f(x)\,dx = 0$

If $x \in (0,2)$, then $F_X(x) = \int_{-\infty}^{x} f(x)\,dx$

$$F_X(x) = \int_{-\infty}^{0} f(x)\,dx + \int_0^x f(x)\,dx$$

$$= \int_{-\infty}^{0} 0\,dx + \int_0^x Kx\,dx = \int_{-\infty}^{0} 0\,dx + \frac{1}{8}\int_0^x x\,dx$$

$$= \left(\frac{x^2}{16}\right)_0^x = \frac{x^2}{16}, \ 0 \le x \le 2$$

If $x \in (2,4)$, then $F_X(x) = \int_{-\infty}^{x} f(x)\,dx$

$$F_X(x) = \int_{-\infty}^{0} f(x)\,dx + \int_0^2 f(x)\,dx + \int_2^x f(x)\,dx$$

$$= \int_{-\infty}^{0} 0\,dx + \int_0^2 Kx\,dx + \int_2^x 2K\,dx$$

$$= \int_0^2 \frac{x}{8}\,dx + \int_2^x \frac{1}{4}\,dx = \left(\frac{x^2}{16}\right)_0^2 + \left(\frac{x}{4}\right)_2^x$$

$$= \frac{1}{4} + \frac{x}{4} - \frac{1}{2}$$

$$= \frac{x}{4} - \frac{4}{16} = \frac{x-1}{4}, \ 2 \le x < 4$$

If $x \in (4,6)$, then $F_X(x) = \int_{-\infty}^{0} 0dx + \int_{0}^{2} Kxdx + \int_{2}^{4} 2Kdx + \int_{4}^{x} k(6-x)\,dx$

$$= \int_0^2 \frac{x}{8}dx + \int_2^4 \frac{1}{4}dx + \int_4^x \frac{1}{8}(6-x)\,dx$$

$$= \left(\frac{x^2}{16}\right)_0^2 + \left(\frac{x}{4}\right)_2^4 + \left(\frac{6x}{8} - \frac{x^2}{16}\right)_4^x$$

$$= \frac{1}{4} + 1 - \frac{1}{2} + \frac{6x}{8} - \frac{x^2}{16} - 3 + 1$$

$$= \frac{4 + 16 - 8 + 12x - x^2 - 48 + 16}{16}$$

$$= \frac{-x^2 + 12x - 20}{16}, \ 4 \le x \le 6$$

If $x > 6$, then $F_X(x) = \int_{-\infty}^{0} 0dx + \int_{0}^{2} Kxdx + \int_{2}^{4} 2Kdx + \int_{4}^{6} k(6-x)\,dx + \int_{6}^{\infty} 0dx$

$$= 1, \ x \ge 6$$

$$\therefore F_X(x) = \begin{cases} 0 & ; x \le 0 \\ \frac{x^2}{16} & ; 0 \le x \le 2 \\ \frac{1}{4}(x-1) & ; 2 \le x \le 4 \\ \frac{-1}{16}(20 - 12x + x^2) & ; 4 \le x \le 6 \\ 1 & ; x \ge 6 \end{cases}$$

Problem18. (a). A random variable X has density function $f(x) = \begin{cases} \frac{K}{1+x^2}, & -\infty < x < \infty \\ 0, & Otherwise \end{cases}$. Determine K and the distribution functions. Evaluate the probability $P(x \ge 0)$.

(b). A random variable X has the P.d.f $f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & Otherwise \end{cases}$

Find (i) $P\left(X < \frac{1}{2}\right)$ (ii) $P\left(\frac{1}{4} < x < \frac{1}{2}\right)$ (iii) $P\left(X > \frac{3}{4} / X > \frac{1}{2}\right)$

Solution (a):

Since $\int_{-\infty}^{\infty} F(x)dx = 1$

$$\int_{-\infty}^{\infty} \frac{K}{1+x^2}dx = 1$$

$$K\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 1$$

$$K\left(\tan^{-1} x\right)_{-\infty}^{\infty} = 1$$

$$K\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 1$$

$$K\pi = 1$$

$$K = \frac{1}{\pi}$$

$$F_X(x) = \int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{x} \frac{K}{1+x^2}dx$$

$$= \frac{1}{\pi}\left(\tan^{-1} x\right)_{-\infty}^{x}$$

$$= \frac{1}{\pi}\left[\tan^{-1} x - \left(-\frac{\pi}{2}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} + \tan^{-1} x\right], -\infty < x < \infty$$

$$P(X \geq 0) = \frac{1}{\pi}\int_0^{\infty} \frac{dx}{1+x^2} = \frac{1}{\pi}\left(\tan^{-1} x\right)_0^{\infty}$$

$$= \frac{1}{\pi}\left(\frac{\pi}{2} - \tan^{-1} 0\right) = \frac{1}{2}.$$

Solution (b):

(i) $P\left(x < \frac{1}{2}\right) = \int_0^{1/2} f(x)\,dx = \int_0^{1/2} 2xdx = 2\left(\frac{x^2}{2}\right)_0^{1/2} = \frac{2\times 1}{8} = \frac{1}{4}$

(ii) $P\left(\frac{1}{4} < x < \frac{1}{2}\right) = \int_{1/4}^{1/2} f(x)\,dx = \int_{1/4}^{1/2} 2xdx = 2\left(\frac{x^2}{2}\right)_{1/4}^{1/2}$

$$= 2\left(\frac{1}{8} - \frac{1}{32}\right) = \left(\frac{1}{4} - \frac{1}{16}\right) = \frac{3}{16}.$$

(iii) $P\left(X > \frac{3}{4} / X > \frac{1}{2}\right) = \dfrac{P\left(X > \frac{3}{4} \cap X > \frac{1}{2}\right)}{P\left(X > \frac{1}{2}\right)} = \dfrac{P\left(X > \frac{3}{4}\right)}{P\left(X > \frac{1}{2}\right)}$

$$P\left(X > \frac{3}{4}\right) = \int_{3/4}^{1} f(x)\,dx = \int_{3/4}^{1} 2x\,dx = 2\left(\frac{x^2}{2}\right)_{3/4}^{1} = 1 - \frac{9}{16} = \frac{7}{16}$$

$$P\left(X > \frac{1}{2}\right) = \int_{1/2}^{1} f(x)\,dx = \int_{1/2}^{1} 2x\,dx = 2\left(\frac{x^2}{2}\right)_{1/2}^{1} = 1 - \frac{1}{4} = \frac{3}{4}$$

$$P\left(X > \frac{3}{4} / X > \frac{1}{2}\right) = \frac{\frac{7}{16}}{\frac{3}{4}} = \frac{7}{16} \times \frac{4}{3} = \frac{7}{12}.$$

Problem 19.(a).If X has the probability density function $f(x) = \begin{cases} Ke^{-3x}, & x > 0 \\ 0 & , otherwise \end{cases}$ find K, $P\left[0.5 \le X \le 1\right]$ and the mean of X.

(b).Find the moment generating function for the distribution whose p.d.f is $f(x) = \lambda e^{-\lambda x},\ x > 0$ and hence find its mean and variance.

Solution:

Since $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^{\infty} Ke^{-3x}dx = 1$$

$$K\left[\frac{e^{-3x}}{-3}\right]_0^{\infty} = 1$$

$$\frac{K}{3} = 1$$

$$K = 3$$

$$P(0.5 \le X \le 1) = \int_{0.5}^{1} f(x)\,dx = 3\int_{0.5}^{1} e^{-3x}dx = 3\frac{e^{-3} - e^{-1.5}}{-3} = \left[e^{-1.5} - e^{-3}\right]$$

Mean of $X = E(x) = \int_0^{\infty} xf(x)\,dx = 3\int_0^{\infty} xe^{-3x}dx$

$$= 3\left[x\left(\frac{-e^{-3x}}{3}\right) - 1\left(\frac{e^{-3x}}{9}\right)\right]_0^{\infty} = \frac{3\times 1}{9} = \frac{1}{3}$$

Hence the mean of $X = E(X) = \frac{1}{3}$

$$M_X(t) = E\left(e^{tx}\right) = \int_0^{\infty} e^{tx} f(x)\,dx = \int_0^{\infty} \lambda e^{-\lambda x} e^{tx} dx$$

$$= \lambda \int_0^{\infty} e^{-x(\lambda - t)} dx$$

$$= \lambda\left[\frac{e^{-x(\lambda-t)}}{-(\lambda-t)}\right]_0^{\infty} = \frac{\lambda}{\lambda-t}$$

$$\text{Mean} = \mu_1' = \left[\frac{d}{dt}M_X(t)\right]_{t=0} = \left[\frac{\lambda}{(\lambda-t)^2}\right]_{t=0} = \frac{1}{\lambda}$$

$$\mu_2' = \left[\frac{d^2}{dt^2}M_X(t)\right]_{t=0} = \left[\frac{\lambda(2)}{(\lambda-t)^3}\right]_{t=0} = \frac{2}{\lambda^2}$$

$$\text{Variance} = \mu_2' - \left(\mu_1'\right)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

Problem 20. (a). If the continuous random variable X has ray Leigh density $F(x) = \left(\frac{x}{\alpha^2} e^{-\frac{x^2}{2\alpha^2}}\right) U(x)$ find $E(x^n)$ and deduce the values of $E(X)$ and $Var(X)$.

(b). Let the random variable X have the p.d.f $f(x) = \begin{cases} \frac{1}{2}e^{-\frac{x}{2}} & , x > 0 \\ 0 & , otherwise. \end{cases}$.

Find the moment generating function, mean & variance of X.

Solution:

(a) Here $U(x) = \begin{cases} 1 & if \quad x > 0 \\ 0 & if \quad x \le 0 \end{cases}$

$$E(x^n) = \int_0^{\infty} x^n f(x)dx$$

$$= \int_0^{\infty} x^n \frac{x}{\alpha^2} e^{\frac{-x^2}{2\alpha^2}} dx$$

Put $\frac{x^2}{2\alpha^2} = t$, $\quad x = 0,\ t = 0$

$\frac{x}{\alpha^2}dx = dt \quad x = \alpha, t = \infty$

$$= \int_0^{\infty} \left(2\alpha^2 t\right)^{n/2} e^{-t}dt \qquad \left[\because x = \sqrt{2}\alpha.\sqrt{t}\right]$$

$$= 2^{n/2}\alpha^n \left(\int_0^{\infty} t^{n/2}e^{-t}dt\right)$$

$$E(x^n) = 2^{n/2}\alpha^n \Gamma\left(\frac{n}{2}+1\right) - (1)$$

Putting $n = 1$ in (1) we get

$$E(x) = 2^{1/2}\alpha\,\Gamma\left(\frac{3}{2}\right) = \sqrt{2}\alpha\,\Gamma\left(\frac{1}{2}+1\right)$$

$$= \sqrt{2}\alpha \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{\alpha}{\sqrt{2}}\sqrt{\pi} \quad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

$$\therefore E(x) = \alpha\sqrt{\frac{\pi}{2}}$$

Putting $n = 2$ in (1), we get

$$E(x^2) = 2\alpha^2\Gamma(2) = 2\alpha^2 \quad [\because \Gamma(2) = 1]$$

$$\therefore Var(X) = E(X^2) - [E(X)]^2$$

$$= 2\alpha^2 - \alpha^2\frac{\pi}{2}$$

$$= \left(2 - \frac{\pi}{2}\right)\alpha^2 = \left(\frac{4-\pi}{2}\right)\alpha^2$$

(b) $$M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \int_0^{\infty} e^{tx}\frac{1}{2}e^{-x/2}dx$$

$$= \frac{1}{2}\int_0^{\infty} e^{\left(t-\frac{1}{2}\right)x}dx = \frac{1}{2}\left[\frac{e^{\left(t-\frac{1}{2}\right)x}}{\left(t-\frac{1}{2}\right)}\right]_0^{\infty} = \frac{1}{1-2t}, \; if\; t < \frac{1}{2}.$$

$$E(X) = \left[\frac{d}{dt}M_X(t)\right]_{t=0} = \left[\frac{2}{(1-2t)^2}\right]_{t=0} = 2$$

$$E(X^2) = \left[\frac{d^2}{dt^2}M_X(t)\right]_{t=0} = \left[\frac{8}{(1-2t)^3}\right]_{t=0} = 8$$

$$Var(X) = E(X^2) - [E(X)]^2 = 8 - 4 = 4.$$

Problem 21. (a).The elementary probability law of a continues random variable is $f(x) = y_0 e^{-b(x-a)},\; a \le x \le \infty,\; b > 0$ where a, b and y_0 are constants. Find y_0 the r[th] moment about point $x = a$ and also find the mean and variance.

(b).The first four moments of a distribution about $x = 4$ are 1,4,10 and 45 respectively. Show that the mean is 5, variance is 3, $\mu_3 = 0$ and $\mu_4 = 26$.

Solution:

Since the total probability is unity,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$y_0\int_0^{\infty} e^{-b(x-a)}dx = 1$$

$$y_0\left[\frac{e^{-b(x-a)}}{-b}\right]_a^{\infty} = 1$$

$$y_0\left(\frac{1}{b}\right) = 1$$

$$y_0 = b.$$

μ_r' (r[th] moment about the point $x = a$) = $\int_{-\infty}^{\infty}(x-a)^r f(x)\,dx$

$$= b\int_a^{\infty}(x-a)^r e^{-b(x-a)}dx$$

Put $x - a = t$, $dx = dt$, when $x = a, t = 0$, $x = \infty, t = \infty$

$$= b\int_0^{\infty} t^r e^{-bt} dt$$

$$= b\frac{\Gamma(r+1)}{b^{(r+1)}} = \frac{r!}{b^r}$$

In particular $r = 1$

$$\mu_1' = \frac{1}{b}$$

$$\mu_2' = \frac{2}{b^2}$$

Mean = $a + \mu_1' = a + \frac{1}{b}$

Variance = $\mu_2' - \left(\mu_1'\right)^2$

$$= \frac{2}{b^2} - \frac{1}{b^2} = \frac{1}{b^2}.$$

b) Given $\mu_1' = 1, \mu_2' = 4, \mu_3' = 10, \mu_4' = 45$

$\mu_r' = r^{th}$ moment about to value $x = 4$

Here $A = 4$

Here Mean = $A + \mu_1' = 4 + 1 = 5$

Variance = $\mu_2 = \mu_2' - \left(\mu_1'\right)^2$

$$= 4 - 1 = 3.$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\left(\mu_1'\right)^3$$

$$= 10 - 3(4)(1) + 2(1)^3 = 0$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\left(\mu_1'\right)^2 - 3\left(\mu_1'\right)^4$$

$$= 45 - 4(10)(1) + 6(4)(1)^2 - 3(1)^4$$

$$\mu_4 = 26.$$

Problem 22. (a). A continuous random variable X has the p.d.f $f(x) = kx^2e^{-x}$, $x \geq 0$. Find the r[th] moment of X about the origin. Hence find mean and variance of X.

(b). Find the moment generating function of the random variable X, with probability density function $f(x) = \begin{cases} x & for \quad 0 \leq x < 1 \\ 2 - x & for \quad 1 \leq x < 2 \\ 0 & otherwise \end{cases}$. Also find μ_1', μ_2'.

Solution:

Since $\int_0^\infty Kx^2e^{-x}dx = 1$

$$K\left[x^2\left(\frac{e^{-x}}{-1}\right) - 2x\left(\frac{e^{-x}}{1}\right) + 2\left(\frac{e^{-x}}{-1}\right)\right]_0^\infty = 1$$

$$2K = 1$$

$$K = \frac{1}{2}.$$

$$\mu_r' = \int_0^\infty x^r f(x)dx$$

$$= \frac{1}{2}\int_0^\infty x^{r+2}e^{-x}dx$$

$$= \frac{1}{2}\int_0^\infty e^{-x}x^{(r+3)-1}dx = \frac{(r+2)!}{2}$$

Putting $n = 1$, $\mu_1' = \frac{3!}{2} = 3$

$n = 2$, $\mu_2' = \frac{41}{2} = 12$

$\therefore$ Mean $= \mu_1' = 3$

Variable$= \mu_2' - \left(\mu_1'\right)^2$

i.e. $\mu_2 = 12 - (3)^2 = 12 - 9$

$\therefore \mu_2 = 3.$

(b) $M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$

$$= \int_0^1 e^{tx}xdx + \int_1^2 e^{tx}(2 - x)\,dx$$

$$= \left(\frac{xe^{tx}}{t} - \frac{e^{tx}}{t^2}\right)_0^1 + \left[(2-x)\frac{e^{tx}}{t} - (-1)\frac{e^{tx}}{t^2}\right]_1^2$$

$$= \frac{e^t}{t} - \frac{e^t}{t^2} + \frac{1}{t^2} + \frac{e^{2t}}{t^2} - \frac{e^t}{t} - \frac{e^t}{t^2}$$

$$= \left(\frac{e^t - 1}{t}\right)^2$$

$$= \left[1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots - 1\right]^2$$

$$= \left[1 + \frac{t}{2!} + \frac{t^2}{3!} + \frac{t^3}{4!} + \ldots\right]^2$$

$$\mu_1' = coeff.\ of\ \frac{t}{1!} = 1$$

$$\mu_2' = coeff.\ of\ \frac{t^2}{2!} = \frac{7}{6}.$$

Problem 23. (a).The p.d.f of the r.v. X follows the probability law: $f(x) = \frac{1}{2\theta} e^{-\frac{|x-\theta|}{\theta}}$, $-\infty < x < \infty$. Find the m.g.f of X and also find $E(X)$ and $V(X)$.

(b).Find the moment generating function and r[th] moments for the distribution. Whose p.d.f is $f(x) = Ke^{-x}$, $0 \le x \le \infty$. Find also standard deviation.

Solution:

$$M_X(t) = E\left(e^{tx}\right) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{2\theta} e^{-\frac{|x-\theta|}{\theta}} e^{tx} dx$$

$$= \int_{-\infty}^{\theta} \frac{1}{2\theta} e^{\frac{(x-\theta)}{\theta}} e^{tx} dx + \int_{\theta}^{\infty} \frac{1}{2\theta} e^{\frac{-(x-\theta)}{\theta}} e^{tx} dx$$

$$M_X(t) = \frac{e^{-1}}{2\theta} \int_{-\infty}^{\theta} e^{x\left(t+\frac{1}{\theta}\right)} dx + \frac{e}{2\theta} \int_{\theta}^{\infty} e^{-x\left(\frac{1}{\theta}-t\right)} dx$$

$$= \frac{e^{-1}}{2\theta}\left(\frac{e^{\theta\left(t+\frac{1}{\theta}\right)}}{t+\frac{1}{\theta}}\right) + \frac{e}{2\theta}\left(\frac{e^{-\theta\left(\frac{1}{\theta}-t\right)}}{\frac{1}{\theta}-t}\right)$$

$$= \frac{e^{\theta t}}{2(\theta t+1)} + \frac{e^{\theta t}}{2(1-\theta t)} = \frac{e^{\theta t}}{1-\theta^2 t^2} = e^{\theta t}\left[1-(\theta t)^2\right]^{-1}$$

$$= \left[1 + \theta t + \frac{\theta^2 t^2}{2!} + \ldots\right]\left[1 + \theta^2 t^2 + \theta^4 t^4 + \ldots\right]$$

$$= 1 + \theta t + \frac{3\theta^2 t^2}{\,} + \ldots$$

2!

$$E(X) = \mu_1' = coeff.\ of\ t\ in\ M_X(t) = \theta$$

$$\mu_2' = coeff.\ of\ \frac{t^2}{2!}\ in\ M_X(t) = 3\theta^2$$

$$Var(X) = \mu_2' - (\mu_1')^2 = 3\theta^2 - \theta^2 = 2\theta^2.$$

b)

Total Probability=1

$$\therefore \int_0^\infty k e^{-x} dx = 1$$

$$k\left[\frac{e^{-x}}{-}\right]_0^\infty = 1$$

$$k = 1$$

$$M_X(t) = E\left[e^{tx}\right] = \int_0^\infty e^{tx} e^{-x} dx = \int_0^\infty e^{(t-1)x} dx$$

$$= \left[\frac{e^{(t-1)x}}{t-1}\right]_0^\infty = \frac{1}{1-t},\ t < 1$$

$$= (1-t)^{-1} = 1 + t + t^2 + \ldots + t^r + \ldots\infty$$

$$\mu_1' = coeff.\ of\ \frac{t^2}{r!} = r!$$

When $r = 1,\ \mu_1' = 1! = 1$

$$r = 2,\ \mu_2' = 2! = 2$$

Variance $= \mu_2' - \mu_1' = 2 - 1 = 1$

$\therefore$ Standard deviation=1.

Problem 24. (a). Define Binomial distribution Obtain its m.g.f., mean and variance.

(b). (i).Six dice are thrown 729 times. How many times do you expect atleast 3 dice show 5 or 6 ?

(ii).Six coins are tossed 6400 times. Using the Poisson distribution, what is the approximate probability of getting six heads x times?

Solution:

a) A random variable X said to follow binomial distribution if it assumes only non negative values and its probability mass function is given by $P(X = x) = nC_x p^x q^{n-x},\ x = 0,1,2,\ldots,n\ and\ q = 1 - p.$

M.G.F.of Binomial distribution:-

M.G.F of Binomial Distribution about origin is

$$M_X(t) = E\left[e^{tx}\right] = \sum_{x=0}^{n} e^{tx} P(X = x)$$

$$= \sum_{x=0}^{n} nC_x P^x q^{n-x} e^{tx}$$

$$= \sum_{x=0}^{n} nC_x \left(pe^t\right)^x q^{n-x}$$

$$M_X(t) = \left(q + pe^t\right)^n$$

Mean of Binomial distribution

$$\text{Mean} = E(X) = M_X'(0)$$

$$= \left[n\left(q + pe^t\right)^{n-1} pe^t \right]_{t=0} = np \text{ Since } q + p = 1$$

$$E\left(X^2\right) = M_X''(0)$$

$$= \left[n(n-1)\left(q + pe^t\right)^{n-2}\left(pe^t\right)^2 + npe^t\left(q + pe^t\right)^{n-1} \right]_{t=0}$$

$$E\left(X^2\right) = n(n-1)p^2 + np$$

$$= n^2p^2 + np(1-p) = n^2p^2 + npq$$

$$\text{Variance} = E\left(X^2\right) - \left[E[X]\right]^2 = npq$$

$$Mean = np \,; Variance = npq$$

b) Let X :the number of times the dice shown 5 or 6

$$P[5 \text{ or } 6] = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

$$\therefore P = \frac{1}{3} \text{ and } q = \frac{2}{3}$$

Here $n = 6$

To evaluate the frequency of $X \geq 3$

By Binomial theorem,

$$P[X = r] = 6C_r \left(\frac{1}{3}\right)^r \left(\frac{2}{3}\right)^{6-r} \quad \text{where } r = 0,1,2...6.$$

$$P[X \geq 3] = P(3) + P(4) + P(5) + P(6)$$

$$= 6C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^3 + 6C_4 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^2 + 6C_5 \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right) + 6C_6 \left(\frac{1}{3}\right)^6$$

$$= 0.3196$$

$\therefore$Expected number of times atleast 3 dies to show 5 or 6 = $N \times P[X \geq 3]$

$$= 729 \times 0.3196 = 233.$$

25. (a). A die is cast until 6 appears what is the probability that it must cast more then five times?

(b). Suppose that a trainee soldier shoots a target an independent fashion. If the probability that the target is shot on any one shot is 0.8.

(i) What is the probability that the target would be hit on 6^{th} attempt?

(ii) What is the probability that it takes him less than 5 shots?

Solution:

Probability of getting six $= \frac{1}{6}$

$\therefore p = \frac{1}{6}$ & $q = 1 - \frac{1}{6}$

Let x: No of throws for getting the number 6.By geometric distribution $P[X = x] = q^{x-1}p, x = 1, 2, 3....$

Since 6 can be got either in first, second……throws.

To find $P[X > 5] = 1 - P[X \le 5]$

$$= 1 - \sum_{x=1}^{5}\left(\frac{5}{6}\right)^{x-1} \cdot \frac{1}{6}$$

$$= 1 - \left[\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^3\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^4\left(\frac{1}{6}\right)\right]$$

$$= 1 - \frac{\frac{1}{6}\left[1 - \left(\frac{5}{6}\right)^5\right]}{1 - \frac{5}{6}} = \left(\frac{5}{6}\right)^5 = 0.4019$$

b) Here $p = 0.8, q = 1 - p = 0.2$

$P[X = r] = q^{r-1}p, r = 0,1,2...$

(i) The probability that the target would be hit on the 6th attempt $= P[X = 6]$

$$= (0.2)^5(0.8) = 0.00026$$

(ii) The probability that it takes him less than 5 shots $= P[X < 5]$

$$= \sum_{r=1}^{4} q^{r-1}p = 0.8\sum_{r=1}^{4}(0.2)^{r-1}$$

$$= 0.8[1 + 0.2 + 0.04 + 0.008] = 0.9984$$

Problem 26. (a). State and prove the memoryless property of exponential distribution.

(b). A component has an exponential time to failure distribution with mean of 10,000 hours.

(i). The component has already been in operation for its mean life. What is the probability that it will fail by 15,000 hours?

(ii). At 15,000 hours the component is still in operation. What is the probability that it will operate for another 5000 hours.

Solution:

a) Statement:

If X is exponentially distributed with parameters λ, then for any two positive integers's' and't', $P[x > s + t / x > s] = P[x > t]$

Proof:

The p.d.f of X is $f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \ge 0 \\ 0 & , Otherwise \end{cases}$

$$\therefore P[X > t] = \int_t^{\infty} \lambda e^{-\lambda x}\, dx = \left[-e^{-\lambda x}\right]_t^{\infty} = e^{-\lambda t}$$

$$\therefore P[X > s+t / x > s] = \frac{P[x > s+t \cap x > s]}{P[x > s]}$$

$$= \frac{P[X > s+t]}{P[X > s]} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t}$$

$$= P[x > t]$$

b) Let X denote the time to failure of the component then X has exponential distribution with $Mean = 1000$ hours.

$$\therefore \frac{1}{\lambda} = 10,000 \Rightarrow \lambda = \frac{1}{10,000}$$

The p.d.f. of X is $f(x) = \begin{cases} \frac{1}{10,000} e^{-\frac{x}{10,000}} & , x \ge 0 \\ 0 & , otherwise \end{cases}$

(i) Probability that the component will fail by 15,000 hours given it has already been in operation for its mean life $= P[x < 15,000 / x > 10,000]$

$$= \frac{P[10,000 < X < 15,000]}{P[X > 10,000]}$$

$$= \frac{\int_{10,000}^{15,000} f(x)\, dx}{\int_{10,000}^{\infty} f(x)\, dx} = \frac{e^{-1} - e^{-1.5}}{e^{-1}}$$

$$= \frac{0.3679 - 0.2231}{0.3679} = 0.3936.$$

(ii) Probability that the component will operate for another 5000 hours given that it is in operational 15,000 hours $= P[X > 20,000 / X > 15,000]$

$$= P[x > 5000] \quad \text{[By memoryless prop]}$$

$$= \int_{5000}^{\infty} f(x)\, dx = e^{-0.5} = 0.6065$$

27. (a). The Daily consumption of milk in a city in excess of 20,000 gallons is approximately distributed as a Gamma variate with parameters $\alpha = 2$ and $\lambda = \frac{1}{10,000}$. The city has a daily stock of 30,000 gallons. What is the probability that the stock is in sufficient on a particular day?

(b). The lifetime (in hours) of a certain piece of equipment is a continuous r.v. having range $0 < x < \infty$ and p.d.f.is $f(x) = \begin{cases} xe^{-kx}, 0 < x < \infty \\ 0 \quad, otherwise \end{cases}$. Determine the constant K and evaluate the probability that the life time exceeds 2 hours.

Solution:

a) Let X be the r.v denoting the daily consumption of milk (is gallons) in a city

Then $Y = X - 20,000$ has Gamma distribution with p.d.f.

$$f(y) = \frac{1}{(10,000)^2 \Gamma(2)} y^{2-1} e^{-\frac{y}{10,000}}, y \geq 0$$

$$f(y) = \frac{ye^{-\frac{y}{10,000}}}{(10,000)^2}, y \geq 0.$$

$\therefore$the daily stock of the city is 30,000 gallons, the required probability that the stock is insufficient on a particular day is given by

$$P[X > 30,000] = P[Y > 10,000]$$

$$= \int_{10,000}^{\infty} g(y)\, dy = \int_{10,000}^{\infty} \frac{ye^{-\frac{y}{10,000}}}{(10,000)^2} dy$$

put $Z = \frac{y}{10,000}$ then $dz = \frac{dy}{10,000}$

$$\therefore P[X > 30,000] = \int_{1}^{\infty} ze^{-z} dz$$

$$= \left[-ze^{-z} - e^{-z}\right]_1^{\infty} = \frac{2}{e}$$

b) Let X the life time of a certain piece of equipment.

Then the p.d.f. $f(x) = \begin{cases} xe^{-kx}, 0 < x < \infty \\ 0 \quad, Otherwise \end{cases}$

To find K, $\int_0^{\infty} f(x)\, dx = 1$

$$\int_0^{\infty} e^{-kx} x^{2-1} dx = 1$$

$$\frac{\Gamma(2)}{K^2} = 1 \Rightarrow K^2 = 1 \;\therefore K = 1$$

$$\therefore f(x) = \begin{cases} xe^{-x}, 0 < x < \infty \\ 0 \quad, Otherwise \end{cases}$$

P[Life time exceeds 2 hours] = $P[X > 2]$

$$= \int_2^{\infty} f(x)\, dx$$

$$= \int_2^{\infty} xe^{-x}dx$$

$$= \left[x\left(-e^{-x}\right) - \left(e^{-x}\right) \right]_2^{\infty}$$

$$= 2e^{-2} + e^{-2} = 3e^{-2} = 0.4060$$

Problem 28. (a). State and prove the additive property of normal distribution.

(b). Prove that "For standard normal distribution $N(0,1)$, $M_X(t) = e^{\frac{t^2}{2}}$.

a) Statement:

If $X_1, X_2, ..., X_n$ are n independent normal random variates with mean (μ_1, σ_1^2), $(\mu_2, \sigma_2^2), ..., (\mu_n, \sigma_n^2)$ then $X_1 + X_2 + ... + X_n$ also a normal random variable with mean $\left(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2\right)$.

Proof:

We know that. $M_{X_1+X_2+...+X_n}(t) = M_{X_1}(t) M_{X_2}(t) ... M_{X_n}(t)$

But $M_{X_i}(t) = e^{\mu_i t + \frac{t^2 \sigma_i^2}{2}}, \ i = 1, 2 ... n$

$$M_{X_1+X_2+...+X_n}(t) = e^{\mu_1 t + \frac{t^2\sigma_1^2}{2}} e^{\mu_2 t + \frac{t^2\sigma_2^2}{2}} ... e^{\mu_n t + \frac{t^2\sigma_n^2}{2}}$$

$$= e^{(\mu_1 + \mu_2 + ... + \mu_n)t + \frac{(\sigma_1^2 + \sigma_2^2 + ... + \sigma_n^2)t^2}{2}}$$

$$= e^{\sum_{i=1}^{n} \mu_i t + \frac{\sum_{i=1}^{n} \sigma_i^2 t^2}{2}}$$

By uniqueness MGF, $X_1 + X_2 + ... + X_n$ follows normal random variable with parameter $\left(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2\right)$.

This proves the property.

b) Moment generating function of Normal distribution

$$= M_X(t) = E\left[e^{tx}\right]$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Put $z = \frac{x-\mu}{\sigma}$ then $dz = \sigma dx, \ -\infty < Z < \infty$

$$\therefore M_X(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\sigma z + \mu) - \frac{z^2}{2}} dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{z^2}{2} - t\sigma z\right)} dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-t\sigma)^2+\left(\frac{\sigma^2 t^2}{2}\right)} dz = \frac{e^{\mu t} e^{\frac{\sigma^2 t^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-t\sigma)^2} dz$$

 the total area under normal curve is unity, we have $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-t\sigma)^2} dz = 1$

Hence $M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$ ∴For standard normal variable $N(0,1)$

$$M_X(t) = e^{\frac{t^2}{2}}$$

Problem 29. (a). The average percentage of marks of candidates in an examination is 45 will a standard deviation of 10 the minimum for a pass is 50%.If 1000 candidates appear for the examination, how many can be expected marks. If it is required, that double that number should pass, what should be the average percentage of marks?

(b). Given that X is normally distribution with mean 10 and probability $P[X > 12] = 0.1587$. What is the probability that X will fall in the interval $(9,11)$.

Solution:

a) Let X be marks of the candidates

Then $X \sim N(42,10^2)$

Let $z = \frac{X-42}{10}$

$P[X > 50] = P[Z > 0.8]$

$= 0.5 - P[0 < z < 0.8] = 0.5 - 0.2881 = 0.2119$

Since 1000 students write the test, nearly 212 students would pass the examination.

If double that number should pass, then the no of passes should be 424.

We have to find z_1, such that $P[Z > z_1] = 0.424$

$\therefore P[0 < z < z_1] = 0.5 - 0.424 = 0.076$

From tables, $z = 0.19$

$\therefore z_1 = \frac{50 - x_1}{10} \Rightarrow x_1 = 50 - 10z_1 = 50 - 1.9 = 48.1$

The average mark should be 48 nearly.

b) Given X is normally distributed with mean $\mu = 10$.

Let $z = \frac{x-\mu}{\sigma}$ be the standard normal variate.

For $X = 12, z = \frac{12-10}{\sigma} \Rightarrow z = \frac{2}{\sigma}$

Put $z_1 = \frac{2}{\sigma}$

Then $P[X > 12] = 0.1587$

$$P[Z > Z_1] = 0.1587$$

$$\therefore 0.5 - p[0 < z < z_1] = 0.1587$$

$$\Rightarrow P[0 < z < z_1] = 0.3413$$

From area table $P[0 < z < 1] = 0.3413$

$$\therefore Z_1 = 1 \Rightarrow \frac{2}{\sigma} = 1$$

To find $P[9 < x < 11]$

For $X = 9,\ z = -\frac{1}{2}$ and $X = 11,\ z = \frac{1}{2}$

$\therefore P[9 < X < 11] = P[-0.5 < z < 0.5] \quad = 2P[0 < z < 0.5] \quad = 2 \times 0.1915 = 0.3830$

Problem. (a). In a normal distribution,31 % of the items are under 45 and 8% are over 64.Find the mean and standard deviation of the distribution.

(b). For a certain distribution the first moment about 10 is 40 and that the 4^{th} moment about 50 is 48, what are the parameters of the distribution.

Solution:

a) Let μ be the mean and σ be the standard deviation.

Then $P[X \le 45] = 0.31$ and $P[X \ge 64] = 0.08$

When $X = 45,\ Z = \frac{45 - \mu}{\sigma} = -z_1$

$\therefore z_1$ is the value of z corresponding to the area $\int_0^{z_1} \varphi(z)dz = 0.19$

$\therefore z_1 = 0.495$

$45 - \mu = -0.495\sigma$ ---(1)

When $X = 64,\ Z = \frac{64 - \mu}{\sigma} = z_2$

$\therefore z_2$ is the value of z corresponding to the area $\int_0^{z_2} \varphi(z)dz = 0.42$

$\therefore z_2 = 1.405$

$64 - \mu = 1.405\sigma$ ---(2)

Solving (1) & (2) We get $\mu = 10$ (approx) & $\sigma = 50$ (approx)

b) Let μ be mean and σ^2 the variance then $\mu_1' = 40$ about A=10

$$\therefore \text{Mean } A + \mu_1' = 10 + 40\ 10+40$$

$$\Rightarrow \mu = 50$$

Also $\mu_4 = 48 \Rightarrow 3\sigma^4 = 48 \Rightarrow \sigma^2 = 4$

$\therefore$ The parameters are Mean $= \mu = 50$ and S.D $= \sigma = 2$.