UNIT –I RANDOM VARIABLES

PART-A

Problem1. X and Y are independent random variables with variance 2 and 3. Find the variance of $3X+4Y$.

Solution:

$$V(3X+4Y) = 9Var(X)+16Var(Y)+24Cov(XY)$$

$$= 9\times 2+16\times 3+0 \quad (\because X \,\&\, Y \text{ are independent } \operatorname{cov}(XY)=0)$$

$$= 18+48 = 66.$$

Problem 2. A Continuous random variable X has a probability density function $F(x)=3x^2$; $0\le x\le 1$. Find 'a' such that $P(x\le a)=P(x>a)$

Solution:

We know that the total probability $=1$

Given $P(X\le a)=P(X>a)=K(say)$

Then $K+K=1$

$$K=\frac{1}{2}$$

ie $P(X\le a)=\frac{1}{2}$ & $P(X>a)=\frac{1}{2}$

Consider $P(X\le a)=\frac{1}{2}$

i.e. $\int_0^a f(x)\,dx=\frac{1}{2}$

$$\int_0^a 3x^2dx=\frac{1}{2}$$

$$3\left(\frac{x^3}{3}\right)_0^a=\frac{1}{2}$$

$$a^3=\frac{1}{2}$$

$$a=\left(\frac{1}{2}\right)^{1/3}.$$

Problem 3. A random variable X has the p.d.f $f(x)$ given by $f(x)=\begin{cases} Cxe^{-x}; & if\ x>0 \\ 0 \quad ; & if\ x\le 0 \end{cases}$

Find the value of C and cumulative density function of X.

Solution:

Since $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{0}^{\infty} Cxe^{-x}dx = 1$$

$$C\left[x\left(-e^{-x}\right)-\left(e^{-x}\right)\right]_0^{\infty} = 1$$

$$C = 1$$

$$\therefore f(x) = \begin{cases} xe^{-x}; & x > 0 \\ 0 & ; x \le 0 \end{cases}$$

$$C.D.F\ F(x) = \int_0^x f(x)dt = \int_0^x te^{-t}dt = \left[-te^{-t} - e^{-t}\right]_0^x = -xe^{-x} - e^{-x} + 1$$

$$= 1 - (1+x)e^{-x}.$$

Problem 4. If a random variable X has the p.d.f $f(x) = \begin{cases} \frac{1}{2}(x+1); & -1 < x < 1 \\ 0 & ; otherwise \end{cases}$.

Find the mean and variance of X .

Solution:

$$\text{Mean} = \int_{-1}^{1} xf(x)\,dx = \frac{1}{2}\int_{-1}^{1} x(x+1)dx = \frac{1}{2}\int_{-1}^{1}\left(x^2 + x\right)dx$$

$$= \frac{1}{2}\left(\frac{x^3}{3} + \frac{x^2}{2}\right)_{-1}^{1} = \frac{1}{3}$$

$$\mu_2' = \int_{-1}^{1} x^2 f(x)dx = \frac{1}{2}\int_{-1}^{1}\left(x^3 + x^2\right)dx = \frac{1}{2}\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\frac{1}{4} + \frac{1}{3} - \frac{1}{4} + \frac{1}{3}\right]$$

$$= \frac{1}{2}\cdot\frac{2}{3} = \frac{1}{3}$$

$$Variance = \mu_2' - \left(\mu_1'\right)^2$$

$$= \frac{1}{3} - \frac{1}{9} = \frac{3-1}{9} = \frac{2}{9}.$$

Problem 5. A random variable X has density function given by $f(x) = \begin{cases} 2e^{-2x}; & x \ge 0 \\ 0 & ; x < 0 \end{cases}$.

Find m.g.f

Solution:

$$M_X(t) = E\left(e^{tx}\right) = \int_0^\infty e^{tx} f(x)\,dx = \int_0^\infty e^{tx}\, 2e^{-2x}\,dx$$

$$= 2\int_0^\infty e^{(t-2)x}\,dx$$

$$= 2\left[\frac{e^{(t-2)x}}{t-2}\right]_0^\infty = \frac{2}{2-t}, t < 2.$$

Problem 6. Criticise the following statement: "The mean of a Poisson distribution is 5 while the standard deviation is 4".

Solution: For a Poisson distribution mean and variance are same. Hence this statement is not true.

Problem 7. Comment the following: "The mean of a binomial distribution is 3 and variance is 4

Solution:

In binomial distribution, mean >variance but Variance < Mean

Since Variance = 4 &Mean = 3, the given statement is wrong.

Problem8. If X and Y are independent binomial variates $B\left(5, \frac{1}{2}\right)$ and $B\left(7, \frac{1}{2}\right)$ find $P[X+Y=3]$

Solution:

$X+Y$ is also a binomial variate with parameters $n_1 + n_2 = 12$ & $p = \frac{1}{2}$

$$\therefore P[X+Y=3] = 12C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^9$$

$$= \frac{55}{2^{10}}$$

Problem 9. If X is uniformly distributed with Mean1and Variance $\frac{4}{3}$, find $P[X>0]$

Solution:

If X is uniformly distributed over (a, b), then

$$E(X) = \frac{b+a}{2} \text{ and } V(X) = \frac{(b-a)^2}{12}$$

$$\therefore \frac{b+a}{2} = 1 \Rightarrow a+b = 2$$

$$\Rightarrow \frac{(b-a)^2}{12} = \frac{4}{3} \Rightarrow (b-a)^2 = 16$$

$\Rightarrow a+b = 2$ & $b-a = 4$ We get $b = 3, a = -1$

$\therefore a = -1$ & $b = 3$ and probability density function of x is

$$f(x) = \begin{cases} \frac{1}{4}; -1 < x < 3 \\ 0 \; Otherwise \end{cases}$$

$$P[x < 0] = \int_{-1}^{0} \frac{1}{4} dx = \frac{1}{4}[x]_{-1}^{0} = \frac{1}{4}.$$

Problem 10. State the memoryless property of geometric distribution.

Solution:

If X has a geometric distribution, then for any two positive integer $'m'$ and $'n'$

$$P\left[X > m + n \big/ X > m\right] = P[X > n].$$

Problem 11. X is a normal variate with $mean = 30$ and $S.D = 5$

Find the following $P[26 \le X \le 40]$

Solution:

$$X \sim N(30, 5^2)$$

$$\therefore \mu = 30 \;\&\; \sigma = 5$$

Let $Z = \frac{X - \mu}{\sigma}$ be the standard normal variate

$$P[26 \le X \le 40] = P\left[\frac{26-30}{5} \le Z \le \frac{40-30}{5}\right]$$

$$= P[-0.8 \le Z \le 2] = P[-0.8 \le Z \le 0] + P[0 \le Z \le 2]$$

$$= P[0 \le Z 0.8] + [0 \le z \le 2]$$

$$= 0.2881 + 0.4772 = 0.7653.$$

Problem 12. If X is a $N(2,3)$ Find $P\left[Y \ge \frac{3}{2}\right]$ where $Y + 1 = X$.

Solution:

$$P\left[Y \ge \frac{3}{2}\right] = P\left[X - 1 \ge \frac{3}{2}\right]$$

$$= P[X \ge 2.5] = P[Z \ge 0.17]$$

$$= 0.5 - P[0 \le Z \le 0.17]$$

$$= 0.5 - 0.0675 = 0.4325$$

Problem 13. If the probability is $\frac{1}{4}$ that a man will hit a target what is the chance that he will hit the target for the first time in the 7th trial?

Solution:

The required probability is

$$P[FFFFFFS] = P(F)\,P(F)\,P(F)\,P(F)\,P(F)\,P(F)\,P(S)$$

$$= q^6 p = \left(\frac{3}{4}\right)^6\left(\frac{1}{4}\right) = 0.0445.$$

Hence p = Probability of hitting target and $q = 1 - p$.

Problem 14. A random variable X has an exponential distribution defined by p.d.f. $f(x) = e^{-x}, 0 < x < \infty$. Find the density function of $Y = 3X + 5$.

Solution:

$$y = 3x + 5 \Rightarrow \frac{dy}{dx} = 3 \Rightarrow \frac{dx}{dy} = \frac{1}{3}$$

P.d.f of y $h_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$

$$h_Y(y) = \frac{1}{3}e^{-x}.$$

Using $x = \frac{y-5}{3}$ we get $h_Y(y) = \frac{1}{3}e^{-\left(\frac{y-5}{3}\right)}, y > 5 \ (\because x > 0 \Rightarrow y > 5)$

Problem 15. If X is a normal variable with zero mean and variance σ^2, Find the p.d.f of $y = e^{-x}$

Solution:

Given $Y = e^{-x}$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2\sigma^2}x^2}$$

$$h_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2\sigma^2}(\log y)^2} \times \frac{1}{y}.$$

PART-B

Problem 16. A random variable X has the following probability function:
Values of X,

X	:	0	1	2	3	4	5	6	7
$P(X)$	:	0	K	$2K$	$2K$	$3K$	K^2	$2K^2$	$7K^2+K$

Find (i) K, (ii) Evaluate $P(X<6)$, $P(X\geq 6)$ and $P(0<X<5)$

(iii). Determine the distribution function of X.

(iv). $P(1.5 < X < 4.5 / X > 2)$

(v). $E(3x-4)$, $Var(3x-4)$

Solution(i):

Since $\sum_{x=0}^{7} P(X) = 1$,

$K + 2K + 2K + 3K + K^2 + 2K^2 + 7K^2 + K = 1$ $10K^2+9K-1=0$

$$K = \frac{1}{10} \quad or \quad K = -1$$

As $P(X)$ cannot be negative $K = \frac{1}{10}$

Solution(ii):

$$P(X<6) = P(X=0)+P(X=1)+\ldots+P(X=5)$$
$$= \frac{1}{10}+\frac{2}{10}+\frac{2}{10}+\frac{3}{10}+\frac{1}{100}+\ldots = \frac{81}{100}$$

Now $P(X \geq 6) = 1 - P(X<6)$

$$= 1-\frac{81}{100} = \frac{19}{100}$$

Now $P(0<X<5) = P(X=1)+P(X=2)+P(X=3) = P(X=4)$

$$= K+2K+2K+3K$$
$$= 8K = \frac{8}{10} = \frac{4}{5}$$

Solution(iii):

The distribution of X is given by $F_X(x)$ defined by

$$F_X(x) = P(X \leq x)$$

X	:	0	1	2	3	4	5	6	7
$F_X(x)$	:	0	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{5}{10}$	$\frac{4}{5}$	$\frac{81}{100}$	$\frac{83}{100}$	1

Problem 17. (a) If $P(x) = \begin{cases} \frac{x}{15} & ; x = 1,2,3,4,5 \\ 0 & ; elsewhere \end{cases}$

Find (i) $P\{X = 1 or 2\}$ and (ii) $P\{1/2 < X < 5/2 \, / \, x > 1\}$

(b) X is a continuous random variable with pdf given by

$$F(X) = \begin{cases} Kx & in \quad 0 \leq x \leq 2 \\ 2K & in \quad 2 \leq x \leq 4 \\ 6K - Kx & in \quad 4 \leq x \leq 6 \\ 0 & elsewhere \end{cases}$$

Find the value of K and also the cdf $F_X(x)$.

Solution:

(a) i) $P(X = 1 or 2) = P(X=1)+P(X=2)$

$$= \frac{1}{15}+\frac{2}{15} = \frac{3}{15} = \frac{1}{5}$$

ii) $$P\left(\frac{1}{2} < X < \frac{5}{2} \, / \, x > 1\right) = \frac{P\left\{\left(\frac{1}{2} < X < \frac{5}{2}\right) \cap (X > 1)\right\}}{P(X>1)}$$

$$= \frac{P\{(X = 1 or 2) \cap (X > 1)\}}{P(X>1)}$$

$$= \frac{P(X=2)}{1-P(X=1)}$$

$$= \frac{2/15}{1-(1/15)} = \frac{2/15}{14/15} = \frac{2}{14} = \frac{1}{7}.$$

Since $\int_{-\infty}^{\infty} F(x)\,dx = 1$

$$\int_0^2 Kx\,dx + \int_2^4 2K\,dx + \int_4^6 (6k - kx)\,dx = 1$$

$$K\left[\left(\frac{x^2}{2}\right)_0^2 + (2x)_2^4 + \left(6x - \frac{x^2}{2}\right)_4^6\right] = 1$$

$$K[\cancel{2} + \cancel{8} - 4 + 36 - 18 - 24 + 8] = 1$$

$$8K = 1 \quad K = \frac{1}{8}$$

We know that $F_X(x) = \int_{-\infty}^{x} f(x)\,dx$

If $x < 0$, then $F_X(x) = \int_{-\infty}^{x} f(x)\,dx = 0$

If $x \in (0, 2)$, then $F_X(x) = \int_{-\infty}^{x} f(x)\,dx$

$$F_X(x) = \int_{-\infty}^{0} f(x)\,dx + \int_0^x f(x)\,dx$$

$$= \int_{-\infty}^{0} 0\,dx + \int_0^x Kx\,dx = \int_{-\infty}^{0} 0\,dx + \frac{1}{8}\int_0^x x\,dx$$

$$= \left(\frac{x^2}{16}\right)_0^x = \frac{x^2}{16}, \; 0 \le x \le 2$$

If $x \in (2, 4)$, then $F_X(x) = \int_{-\infty}^{x} f(x)\,dx$

$$F_X(x) = \int_{-\infty}^{0} f(x)\,dx + \int_0^2 f(x)\,dx + \int_2^x f(x)\,dx$$

$$= \int_{-\infty}^{0} 0\,dx + \int_0^2 Kx\,dx + \int_2^x 2K\,dx$$

$$= \int_0^2 \frac{x}{8}\,dx + \int_2^x \frac{1}{4}\,dx = \left(\frac{x^2}{16}\right)_0^2 + \left(\frac{x}{4}\right)_2^x$$

$$= \frac{1}{4} + \frac{x}{4} - \frac{1}{2}$$

$$= \frac{x}{4} - \frac{4}{16} = \frac{x-1}{4}, \ 2 \le x < 4$$

If $x \in (4, 6)$, then $F_X(x) = \int_{-\infty}^{0} 0\,dx + \int_{0}^{2} Kx\,dx + \int_{2}^{4} 2K\,dx + \int_{4}^{x} k(6-x)\,dx$

$$= \int_{0}^{2} \frac{x}{8}dx + \int_{2}^{4} \frac{1}{4}dx + \int_{4}^{x} \frac{1}{8}(6-x)\,dx$$

$$= \left(\frac{x^2}{16}\right)_0^2 + \left(\frac{x}{4}\right)_2^4 + \left(\frac{6x}{8} - \frac{x^2}{16}\right)_4^x$$

$$= \frac{1}{4} + 1 - \frac{1}{2} + \frac{6x}{8} - \frac{x^2}{16} - 3 + 1$$

$$= \frac{4 + 16 - 8 + 12x - x^2 - 48 + 16}{16}$$

$$= \frac{-x^2 + 12x - 20}{16}, \ 4 \le x \le 6$$

If $x > 6$, then $F_X(x) = \int_{-\infty}^{0} 0\,dx + \int_{0}^{2} Kx\,dx + \int_{2}^{4} 2K\,dx + \int_{4}^{6} k(6-x)\,dx + \int_{6}^{\infty} 0\,dx$

$$= 1, \ x \ge 6$$

$$\therefore F_X(x) = \begin{cases} 0 & ; x \le 0 \\ \dfrac{x^2}{16} & ; 0 \le x \le 2 \\ \dfrac{1}{4}(x-1) & ; 2 \le x \le 4 \\ \dfrac{-1}{16}(20 - 12x + x^2) & ; 4 \le x \le 6 \\ 1 & ; x \ge 6 \end{cases}$$

Problem18. (a). A random variable X has density function $f(x) = \begin{cases} \dfrac{K}{1+x^2}, & -\infty < x < \infty \\ 0, & Otherwise \end{cases}$. Determine K and the distribution functions. Evaluate the probability $P(x \ge 0)$.

(b). A random variable X has the P.d.f $f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & Otherwise \end{cases}$

Find (i) $P\left(X < \frac{1}{2}\right)$ (ii) $P\left(\frac{1}{4} < x < \frac{1}{2}\right)$ (iii) $P\left(X > \frac{3}{4} / X > \frac{1}{2}\right)$

Solution (a):

Since $\int_{-\infty}^{\infty} F(x)\,dx = 1$

$$\int_{-\infty}^{\infty} \frac{K}{1+x^2}\,dx = 1$$

$$K\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 1$$

$$K\left(\tan^{-1} x\right)_{-\infty}^{\infty} = 1$$

$$K\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 1$$

$$K\pi = 1$$

$$K = \frac{1}{\pi}$$

$$F_X(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{x} \frac{K}{1+x^2}\,dx$$

$$= \frac{1}{\pi}\left(\tan^{-1} x\right)_{-\infty}^{x}$$

$$= \frac{1}{\pi}\left[\tan^{-1} x - \left(-\frac{\pi}{2}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} + \tan^{-1} x\right], \quad -\infty < x < \infty$$

$$P(X \geq 0) = \frac{1}{\pi}\int_{0}^{\infty} \frac{dx}{1+x^2} = \frac{1}{\pi}\left(\tan^{-1} x\right)_{0}^{\infty}$$

$$= \frac{1}{\pi}\left(\frac{\pi}{2} - \tan^{-1} 0\right) = \frac{1}{2}.$$

Solution (b):

(i) $P\left(x < \frac{1}{2}\right) = \int_{0}^{1/2} f(x)\,dx = \int_{0}^{1/2} 2x\,dx = 2\left(\frac{x^2}{2}\right)_{0}^{1/2} = \frac{2\times 1}{8} = \frac{1}{4}$

(ii) $P\left(\frac{1}{4} < x < \frac{1}{2}\right) = \int_{1/4}^{1/2} f(x)\,dx = \int_{1/4}^{1/2} 2x\,dx = 2\left(\frac{x^2}{2}\right)_{1/4}^{1/2}$

$$= 2\left(\frac{1}{8} - \frac{1}{32}\right) = \left(\frac{1}{4} - \frac{1}{16}\right) = \frac{3}{16}.$$

(iii) $P\left(X > \frac{3}{4} / X > \frac{1}{2}\right) = \dfrac{P\left(X > \frac{3}{4} \cap X > \frac{1}{2}\right)}{P\left(X > \frac{1}{2}\right)} = \dfrac{P\left(X > \frac{3}{4}\right)}{P\left(X > \frac{1}{2}\right)}$

$$P\left(X > \frac{3}{4}\right) = \int_{3/4}^{1} f(x)\,dx = \int_{3/4}^{1} 2x\,dx = 2\left(\frac{x^2}{2}\right)_{3/4}^{1} = 1 - \frac{9}{16} = \frac{7}{16}$$

$$P\left(X > \frac{1}{2}\right) = \int_{1/2}^{1} f(x)\,dx = \int_{1/2}^{1} 2x\,dx = 2\left(\frac{x^2}{2}\right)_{1/2}^{1} = 1 - \frac{1}{4} = \frac{3}{4}$$

$$P\left(X > \frac{3}{4} \Big/ X > \frac{1}{2}\right) = \frac{\frac{7}{16}}{\frac{3}{4}} = \frac{7}{16} \times \frac{4}{3} = \frac{7}{12}$$

Problem 19.(a).If X has the probability density function $f(x) = \begin{cases} Ke^{-3x}, & x > 0 \\ 0 & ,otherwise \end{cases}$ find K, $P[0.5 \le X \le 1]$ and the mean of X.

(b).Find the moment generating function for the distribution whose p.d.f is $f(x) = \lambda e^{-\lambda x}$, $x > 0$ and hence find its mean and variance.

Solution:

Since $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{0}^{\infty} Ke^{-3x}\,dx = 1$$

$$K\left[\frac{e^{-3x}}{-3}\right]_0^{\infty} = 1$$

$$\frac{K}{3} = 1$$

$$K = 3$$

$$P(0.5 \le X \le 1) = \int_{0.5}^{1} f(x)\,dx = 3\int_{0.5}^{1} e^{-3x}\,dx = \cancel{3}\,\frac{e^{-3} - e^{-1.5}}{-\cancel{3}} = \left[e^{-1.5} - e^{-3}\right]$$

Mean of $X = E(x) = \int_0^{\infty} xf(x)\,dx = 3\int_0^{\infty} xe^{-3x}\,dx$

$$= 3\left[x\left(\frac{-e^{-3x}}{3}\right) - (1)\left(\frac{e^{-3x}}{9}\right)\right]_0^{\infty} = \frac{3 \times 1}{9} = \frac{1}{3}$$

Hence the mean of $X = E(X) = \frac{1}{3}$

$$M_X(t) = E\left(e^{tx}\right) = \int_0^{\infty} e^{tx} f(x)\,dx = \int_0^{\infty} \lambda e^{-\lambda x} e^{tx}\,dx$$

$$= \lambda \int_0^{\infty} e^{-x(\lambda - t)}\,dx$$

$$= \lambda\left[\frac{e^{-x(\lambda-t)}}{-(\lambda-t)}\right]_0^{\infty} = \frac{\lambda}{\lambda-t}$$

$$\text{Mean} = \mu_1' = \left[\frac{d}{dt}M_X(t)\right]_{t=0} = \left[\frac{\lambda}{(\lambda-t)^2}\right]_{t=0} = \frac{1}{\lambda}$$

$$\mu_2' = \left[\frac{d^2}{dt^2}M_X(t)\right]_{t=0} = \left[\frac{\lambda(2)}{(\lambda-t)^3}\right]_{t=0} = \frac{2}{\lambda^2}$$

$$\text{Variance} = \mu_2' - \left(\mu_1'\right)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

Problem 20. (a). If the continuous random variable X has ray Leigh density $F(x) = \left(\frac{x}{\alpha^2} e^{-\frac{x^2}{2\alpha^2}}\right) U(x)$ find $E(x^n)$ and deduce the values of $E(X)$ and $Var(X)$.

(b). Let the random variable X have the p.d.f $f(x) = \begin{cases} \frac{1}{2}e^{-\frac{x}{2}} & , x > 0 \\ 0 & , otherwise. \end{cases}$.

Find the moment generating function, mean & variance of X.

Solution:

(a) Here $U(x) = \begin{cases} 1 & if \quad x > 0 \\ 0 & if \quad x \le 0 \end{cases}$

$$E\left(x^n\right) = \int_0^{\infty} x^n f(x) dx$$

$$= \int_0^{\infty} x^n \frac{x}{\alpha^2} e^{\frac{-x^2}{2\alpha^2}} dx$$

Put $\frac{x^2}{2\alpha^2} = t$, $\quad x = 0,\ t = 0$

$\frac{x}{\alpha^2} dx = dt$ $\quad x = \infty,\ t = \infty$

$$= \int_0^{\infty} \left(2\alpha^2 t\right)^{n/2} e^{-t} dt \qquad \left[\because x = \sqrt{2}\alpha\sqrt{t}\right]$$

$$= 2^{n/2}\alpha^n \int_0^{\infty} t^{n/2} e^{-t} dt$$

$$E\left(x^n\right) = 2^{n/2}\alpha^n \Gamma\left(\frac{n}{2} + 1\right) \quad (1)$$

Putting $n = 1$ in (1) we get $E(x) = 2^{1/2}\alpha\left(\frac{3}{2}\right) = \sqrt{2}\alpha\Gamma\left(\frac{1}{2} + 1\right)$

$$= \sqrt{2}\,\alpha\,\Gamma\left(\frac{1}{2}+1\right)$$

$$= \frac{\alpha}{\sqrt{2}}\sqrt{\pi} \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

$$\therefore E(x) = \alpha\sqrt{\frac{\pi}{2}}$$

Putting $n = 2$ in (1), we get

$$E(x^2) = 2\alpha^2\Gamma(2) = 2\alpha^2 \qquad [\because \Gamma(2) = 1]$$

$$\therefore Var(X) = E(X^2) - [E(X)]^2$$

$$= 2\alpha^2 - \alpha^2\frac{\pi}{2}$$

$$= \left(2 - \frac{\pi}{2}\right)\alpha^2 = \left(\frac{4-\pi}{2}\right)\alpha^2$$

(b) $$M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \int_0^{\infty} e^{tx}\frac{1}{2}e^{-x/2}\,dx$$

$$= \frac{1}{2}\int_0^{\infty} e^{\left(t-\frac{1}{2}\right)x}\,dx = \frac{1}{2}\left[\frac{e^{\left(t-\frac{1}{2}\right)x}}{\left(t-\frac{1}{2}\right)}\right]_0^{\infty} = \frac{1}{1-2t}, \text{ if } t < \frac{1}{2}.$$

$$E(X) = \left[\frac{d}{dt}M_X(t)\right]_{t=0} = \left[\frac{2}{(1-2t)^2}\right]_{t=0} = 2$$

$$E(X^2) = \left[\frac{d^2}{dt^2}M_X(t)\right]_{t=0} = \left[\frac{8}{(1-2t)^3}\right]_{t=0} = 8$$

$$Var(X) = E(X^2) - [E(X)]^2 = 8 - 4 = 4.$$

Problem 21. (a).The elementary probability law of a continues random variable is $f(x) = y_0 e^{-b(x-a)},\ a \le x \le \infty,\ b > 0$ where a, b and y_0 are constants. Find y_0 the r[th] moment about point $x = a$ and also find the mean and variance.

(b).The first four moments of a distribution about $x = 4$ are 1,4,10 and 45 respectively. Show that the mean is 5, variance is 3, $\mu_3 = 0$ and $\mu_4 = 26$.

Solution:

Since the total probability is unity,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$y_0\int_0^{\infty} e^{-b(x-a)}\,dx = 1$$

$$y_0\left[\frac{e^{-b(x-a)}}{-b}\right]_a^{\infty}=1$$

$$y_0\left(\frac{1}{b}\right)=1$$

$$\Rightarrow y_0=b.$$

μ_r' (r^{th} moment about the point $x=a$) $=\int_{-\infty}^{\infty}(x-a)^r f(x)\,dx$

$$=b\int_a^{\infty}(x-a)^r e^{-b(x-a)}dx$$

Put $x-a=t$, $dx=dt$, when $x=a, t=0$, $x=\infty, t=\infty$

$$=b\int_0^{\infty}t^r e^{-bt}dt$$

$$=b\frac{\Gamma(r+1)}{b^{(r+1)}}=\frac{r!}{b^r}$$

In particular $r=1$

$$\mu_1'=\frac{1}{b}$$

$$\mu_2'=\frac{2}{b^2}$$

Mean $=a+\mu_1'=a+\frac{1}{b}$

Variance $=\mu_2'-\left(\mu_1'\right)^2$

$$=\frac{2}{b^2}-\frac{1}{b^2}=\frac{1}{b^2}.$$

b) Given $\mu_1'=1, \mu_2'=4, \mu_3'=10, \mu_4'=45$

μ_r' = r^{th} moment about to value $x=4$

Here $A=4$

Here Mean $=A+\mu_1'=4+1=5$

Variance $=\mu_2=\mu_2'-\left(\mu_1'\right)^2$

$$=4-1=3.$$

$$\mu_3=\mu_3'-3\mu_1'\mu_2'+2\left(\mu_1'\right)^3$$

$$=10-3(4)(1)+2(1)^3=0$$

$$\mu_4=\mu_4'-4\mu_1'\mu_3'+6\mu_1'^2\mu_2'-3\left(\mu_1'\right)^4$$

$= 45 - 4(10)(1) + 6(4)(1)^2 - 3(1)^4$

$\mu_4 = 26.$

Problem 22. (a). A continuous random variable X has the p.d.f $f(x) = kx^2 e^{-x}, x \geq 0$. Find the r^th moment of X about the origin. Hence find mean and variance of X.

(b). Find the moment generating function of the random variable X, with probability density function $f(x) = \begin{cases} x & for \quad 0 \leq x < 1 \\ 2 - x & for \quad 1 \leq x < 2 \\ 0 & otherwise \end{cases}$. Also find μ_1', μ_2'.

Solution:

Since $\int_0^{\infty} Kx^2 e^{-x} dx = 1$

$$K\left[x^2\left(\frac{e^{-x}}{-1}\right) - 2x\left(\frac{e^{-x}}{1}\right) + 2\left(\frac{e^{-x}}{-1}\right)\right]_0^{\infty} = 1$$

$$2K = 1$$

$$K = \frac{1}{2}.$$

$$\mu_r' = \int_0^{\infty} x^r f(x) dx$$

$$= \frac{1}{2}\int_0^{\infty} x^{r+2} e^{-x} dx$$

$$= \frac{1}{2}\int_0^{\infty} e^{-x} x^{(r+3)-1} dx = \frac{(r+2)!}{2}$$

Putting $n = 1$, $\mu_1' = \frac{3!}{2} = 3$

$n = 2$, $\mu_2' = \frac{41}{2} = 12$

$\therefore$ Mean $= \mu_1' = 3$

Variable$= \mu_2' - (\mu_1')^2$

i.e. $\mu_2 = 12 - (3)^2 = 12 - 9$

$\therefore \mu_2 = 3.$

(b) $M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$

$$= \int_0^1 e^{tx} x dx + \int_1^2 e^{tx}(2 - x) dx$$

$$= \left[\left(\frac{xe^{tx}}{t} - \frac{e^{tx}}{t^2}\right)\right]_0^1 + \left[(2-x)\frac{e^{tx}}{t} - (-1)\frac{e^{tx}}{t^2}\right]_1^2$$

$$= \frac{e^t}{t} - \frac{e^t}{t^2} + \frac{1}{t^2} + \frac{e^{2t}}{t^2} - \frac{e^t}{t} - \frac{e^t}{t^2}$$

$$= \left(\frac{e^t - 1}{t}\right)^2$$

$$= \left[\frac{1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots - 1}{t}\right]^2$$

$$= \left[1 + \frac{t}{2!} + \frac{t^2}{3!} + \frac{t^3}{4!} + \dots\right]^2$$

$$\mu_1' = coeff.\ of\ \frac{t}{1!} = 1$$

$$\mu_2' = coeff.\ of\ \frac{t^2}{2!} = \frac{7}{6}.$$

Problem 23. (a).The p.d.f of the r.v. X follows the probability law: $f(x) = \frac{1}{2\theta} e^{-\frac{|x-\theta|}{\theta}}$, $-\infty < x < \infty$. Find the m.g.f of X and also find $E(X)$ and $V(X)$

(b).Find the moment generating function and r^{th} moments for the distribution. Whose p.d.f is $f(x) = Ke^{-x}$, $0 \le x \le \infty$. Find also standard deviation.

Solution:

$$M_X(t) = E\left(e^{tx}\right) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx = \int_{-\infty}^{\infty} \frac{1}{2\theta} e^{-\frac{|x-\theta|}{\theta}} e^{tx} dx$$

$$= \int_{-\infty}^{\theta} \frac{1}{2\theta} e^{\frac{(x-\theta)}{\theta}} e^{tx} dx + \int_{\theta}^{\infty} \frac{1}{2\theta} e^{\frac{-(x-\theta)}{\theta}} e^{tx} dx$$

$$M_X(t) = \frac{e^{-1}}{2\theta} \int_{-\infty}^{\theta} e^{x\left(t + \frac{1}{\theta}\right)} dx + \frac{e}{2\theta} \int_{\theta}^{\infty} e^{-x\left(\frac{1}{\theta} - t\right)} dx$$

$$= \frac{e^{-1}}{2\theta}\left(\frac{e^{\theta\left(t+\frac{1}{\theta}\right)}}{t + \frac{1}{\theta}}\right) + \frac{e}{2\theta}\left(\frac{e^{-\theta\left(\frac{1}{\theta}-t\right)}}{\frac{1}{\theta} - t}\right)$$

$$= \frac{e^{\theta t}}{2(\theta t + 1)} + \frac{e^{\theta t}}{2(1 - \theta t)} = \frac{e^{\theta t}}{1 - \theta^2 t^2} = e^{\theta t}\left[1 - (\theta t)^2\right]^{-1}$$

$$= \left[1 + \theta t + \frac{\theta^2 t^2}{2!} + \dots\right]\left[1 + \theta^2 t^2 + \theta^4 t^4 + \dots\right]$$

$$= 1 + \theta t + \frac{3\theta^2 t^2}{2} + \dots$$

$$E(X) = \mu_1' = coeff.\ of\ t\ in\ M_X(t) = \theta$$

$$\mu_2' = coeff.\ of\ \frac{t^2}{2!}\ in\ M_X(t) = 3\theta^2$$

$$Var(X) = \mu_2' - \left(\mu_1'\right)^2 = 3\theta^2 - \theta^2 = 2\theta^2.$$

b)

Total Probability=1

$$\therefore \int_0^\infty k e^{-x} dx = 1$$

$$k\left[\frac{e^{-x}}{-1}\right]_0^\infty = 1$$

$$k = 1$$

$$M_X(t) = E\left[e^{tx}\right] = \int_0^\infty e^{tx} e^{-x} dx = \int_0^\infty e^{(t-1)x} dx$$

$$= \left[\frac{e^{(t-1)x}}{t-1}\right]_0^\infty = \frac{1}{1-t},\ t < 1$$

$$= (1-t)^{-1} = 1 + t + t^2 + \dots + t^r + \dots\infty$$

$$\mu_r' = coeff.\ of\ \frac{t^r}{r!} = r!$$

When $r = 1$, $\mu_1' = 1! = 1$

$r = 2$, $\mu_2' = 2! = 2$

Variance $= \mu_2' - \mu_1'^2 = 2 - 1 = 1$

$\therefore$ Standard deviation=1.

Problem 24. (a). Define Binomial distribution Obtain its m.g.f., mean and variance.

(b). (i).Six dice are thrown 729 times. How many times do you expect atleast 3 dice show 5 or 6 ?

(ii).Six coins are tossed 6400 times. Using the Poisson distribution, what is the approximate probability of getting six heads x times?

Solution:

a) A random variable X said to follow binomial distribution if it assumes only non negative values and its probability mass function is given by $P(X = x) = nC_x p^x q^{n-x}$, $x = 0,1,2,\dots,n$ and $q = 1 - p$. M.G.F.of Binomial distribution:-

M.G.F of Binomial Distribution about origin is

$$M_X(t) = E\left[e^{tx}\right] = \sum_{x=0}^{n} e^{tx} P(X = x)$$

$$= \sum_{x=0}^{n} nC_x P^x q^{n-x} e^{tx}$$

UNIT – II TWO DIMENSIONAL RANDOM VARIABLES

Part.A

Problem 1. Let X and Y have joint density function $f(x, y) = 2,\ 0 < x < y < 1$. Find the marginal density function. Find the conditional density function Y given $X = x$.

Solution:

Marginal density function of X is given by

$$f_X(x) = f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_x^1 f(x,y)\,dy = \int_x^1 2\,dy = 2(y)_x^1$$

$$= 2(1-x),\ 0 < x < 1.$$

Marginal density function of Y is given by

$$f_Y(y) = f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_0^y 2\,dx = 2y,\ 0 < y < 1.$$

Conditional distribution function of Y given $X = x$ is $f\left(\frac{y}{x}\right) = \frac{f(x,y)}{f(x)} = \frac{2}{2(1-x)} = \frac{1}{1-x}$.

Problem 2. Verify that the following is a distribution function. $F(x) = \begin{cases} 0 & , x < -a \\ \frac{1}{2}\left(\frac{x}{a}+1\right) & , -a < x < a \\ 1 & , x > a \end{cases}$.

Solution:

$F(x)$ is a distribution function only if $f(x)$ is a density function.

$$f(x) = \frac{d}{dx}[F(x)] = \frac{1}{2a},\quad -a < x < a$$

$$\int_{-\infty}^{\infty} f(x) = 1$$

$$\therefore \int_{-a}^{a} \frac{1}{2a}\,dx = \frac{1}{2a}[x]_{-a}^{a} = \frac{1}{2a}[a - (-a)]$$

$$= \frac{1}{2a}.2a = 1 .$$

Therefore, it is a distribution function.

Problem 3. Prove that $\int_{x_1}^{x_2} f_X(x)\,dx = p(x_1 < x < x_2)$

Solution:

$$\int_{x_1}^{x_2} f_X(x)dx = \left[F_X(x)\right]_{x_1}^{x_2}$$

$$= F_X(x_2) - F_X(x_1)$$

$$= P[X \le x_2] - P[X \le x_1]$$

$$= P[x_1 \le X \le x_2]$$

Problem 4. A continuous random variable X has a probability density function $f(x) = 3x^2$, $0 \le x \le 1$. Find 'a' such that $P(X \le a) = P(X > a)$.

Solution:

Since $P(X \le a) = P(X > a)$, each must be equal to $\frac{1}{2}$ because the probability is always 1.

$$\therefore P(X \le a) = \frac{1}{2}$$

$$\Rightarrow \int_0^a f(x)\,dx = \frac{1}{2}$$

$$\int_0^a 3x^2 dx = \frac{1}{2} \Rightarrow 3\left[\frac{x^3}{3}\right]_0^a = a^3 = \frac{1}{2}.$$

$$\therefore a = \left(\frac{1}{2}\right)^{\frac{1}{3}}$$

Problem5. Suppose that the joint density function

$$f(x,y) = \begin{cases} Ae^{-x-y}, & 0 \le x \le y, 0 \le y \le \infty \\ 0 & , otherwise \end{cases}$$ Determine A.

Solution:

Since $f(x, y)$ is a joint density function

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\,dxdy = 1.$$

$$\Rightarrow \int_0^{\infty}\int_0^{y} Ae^{-x}e^{-y}dxdy = 1$$

$$\Rightarrow A\int_0^\infty e^{-y}\left(\frac{e^{-xy}}{-y}\right)_0^\infty dy = 1$$

$$\Rightarrow A\int_0^\infty \left[e^{-y} - e^{-2y}\right]dy = 1$$

$$\Rightarrow A\left[\frac{e^{-y}}{-1} - \frac{e^{-2y}}{-2}\right]_0^\infty = 1$$

$$\Rightarrow A\left[\frac{1}{2}\right] = 1 \Rightarrow A = 2$$

Problem 6. Examine whether the variables X and Y are independent, whose joint density function is $f(x,y) = xe^{-x(y+1)}, 0 < x, y < \infty$.

Solution:

The marginal probability function of X is

$$f_X(x) = f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^\infty xe^{-x(y+1)}dy$$

$$= x\left[\frac{e^{-x(y+1)}}{-x}\right]_0^\infty = -\left[0 - e^{-x}\right] = e^{-x},$$

The marginal probability function of Y is

$$f_Y(y) = f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^\infty xe^{-x(y+1)}dx$$

$$= x\left\{\left[\frac{e^{-x(y+1)}}{-(y+1)}\right]_0^\infty - \left[\frac{e^{-x(y+1)}}{(y+1)^2}\right]_0^\infty\right\}$$

$$= \frac{1}{(y+1)^2}$$

Here $f(x).f(y) = e^{-x} \times \frac{1}{(1+y)^2} \neq f(x,y)$

$\therefore X$ and Y are not independent.

Problem 7. If X has an exponential distribution with parameter 1. Find the pdf of $y = \sqrt{x}$

Solution:

Since $y = \sqrt{x}$, $x = y^2$

Since X has an exponential distribution with parameter 1, the pdf of X is given by

$$f_X(x) = e^{-x}, x > 0 \qquad \left[f(x) = \lambda e^{-\lambda x}, \lambda = 1\right]$$

$$\therefore f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$$

$$= e^{-x}2y = 2ye^{-y^2}$$

$$f_Y(y) = 2ye^{-\frac{y}{2}},\ y > 0$$

Problem 8. If X is uniformly distributed random variable in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, Find the probability density function of $Y = tanX$.

Solution:

Given $Y = tanX \Rightarrow x = tan^{-1}y$

$$\therefore \frac{dx}{dy} = \frac{1}{1+y^2}$$

Since X is uniformly distribution in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

$$f_X(x) = \frac{1}{b-a} = \frac{1}{\frac{\pi}{2}+\frac{\pi}{2}}$$

$$f_X(x) = \frac{1}{\pi},\ -\frac{\pi}{2} < x < \frac{\pi}{2}$$

Now $f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = \frac{1}{\pi}\left(\frac{1}{1+y^2}\right),\ -\infty < y < \infty$

$$\therefore f_Y(y) = \frac{1}{\pi(1+y^2)},\ -\infty < y < \infty$$

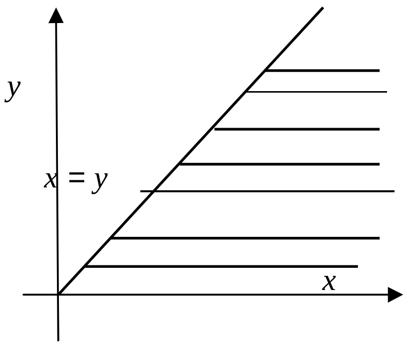
Problem 9. If the Joint probability density function of (x, y) is given by $f(x,y) = 24y(1-x)$, $0 \le y \le x \le 1$ Find $E(XY)$.

Solution:

$$E(xy) = \int_0^1\int_y^1 xyf(x,y)\,dxdy$$

$$= 24\int_0^1\int_y^1 xy^2(1-x)\,dxdy$$

$$= 24\int_0^1 y^2\left[\frac{1}{6} - \frac{y^2}{2} + \frac{y^3}{3}\right]dy = \frac{4}{15}.$$



Problem 10. If X and Y are random Variables, Prove that $Cov(X,Y) = E(XY) - E(X)E(Y)$

Solution:

$$cov(X,Y) = E\left[\left(X - E(X)\right)\left(Y - E(Y)\right)\right]$$

$$= E\left(XY - \overline{X}Y - \overline{Y}X + \overline{X}\,\overline{Y}\right)$$

$$= E(XY) - \overline{X}E(Y) - \overline{Y}E(X) + \overline{X}\,\overline{Y}$$

$$= E(XY) - \overline{X}\,\overline{Y} - \overline{X}\,\overline{Y} + \overline{X}\,\overline{Y}$$

$$= E(XY) - E(X)\,E(Y) \qquad \left[\, E(X) = \bar{X}, E(Y) = \bar{Y} \right]$$

Problem 11. If X and Y are independent random variables prove that $cov(x, y) = 0$

Proof:

$cov(x, y) = E(xy) - E(x)\,E(y)$

But if X and Y are independent then $E(xy) = E(x)\,E(y)$

$cov(x, y) = E(x)E(y) - E(x)E(y)$

$cov(x, y) = 0.$

Problem 12. Write any two properties of regression coefficients.

Solution:

1. Correction coefficients is the geometric mean of regression coefficients
2. If one of the regression coefficients is greater than unity then the other should be less than 1.

$$b_{xy} = r\frac{\sigma_y}{\sigma_x} \text{ and } b_{yx} = r\frac{\sigma_x}{\sigma_y}$$

If $b_{xy} > 1$ then $b_{yx} < 1$.

Problem 13. Write the angle between the regression lines.

Solution: The slopes of the regression lines are

$$m_1 = r\frac{\sigma_y}{\sigma_x}, \quad m_2 = \frac{1}{r}\frac{\sigma_y}{\sigma_x}$$

If θ is the angle between the lines, Then

$$\tan\theta = \frac{\sigma_x\sigma_y}{\sigma_x^2 + \sigma_y^2}\left[\frac{1 - r^2}{r}\right]$$

When $r = 0$, that is when there is no correlation between x and y, $\tan\theta = \infty$ (or) $\theta = \frac{\pi}{2}$ and so the regression lines are perpendicular

When $r = 1$ or $r = -1$, that is when there is a perfect correlation $+ve$ or $-ve$, $\theta = 0$ and so the lines coincide.

Problem 14. State central limit theorem

Solution:

If $X_1, X_2....X_n$ is a sequence of independent random variable $E(X_i) = \mu_i$ and $Var(X_i) = \sigma_i^2$, $i = 1, 2, ... n$ and if $S_n = X_1 + X_2 + ... + X_n$ then under several conditions S_n follows a normal distribution with mean $\mu = \sum_{i=1}^{n} \mu_i$ and variance $\sigma^2 = \sum_{i=1}^{n} \sigma_i^2$ as $n \to \infty$.

Problem 15. i). Two random variables are said to be orthogonal if correlation is zero.

ii). If $X = Y$ then correlation coefficient between them is <u>1.</u>

Part-B

Problem 16. a). The joint probability density function of a bivariate random variable (X, Y) is

$$f_{XY}(x,y) = \begin{cases} k(x+y), & 0 < x < 2,\ 0 < y < 2 \\ 0, & otherwise \end{cases}$$ where $'k'$ is a constant.

i. Find k.

ii. Find the marginal density function of X and Y.

iii. Are X and Y independent?

iv. Find $f_{Y/X}\left(y/x\right)$ and $f_{X/Y}\left(x/y\right)$.

Solution:

(i). Given the joint probability density function of a brivate random variable (X, Y) is

$$f_{XY}(x,y) = \begin{cases} K(x+y), & 0 < x < 2,\ 0 < y < 2 \\ 0, & otherwise \end{cases}$$

$$\text{Here} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dxdy = 1 \Rightarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(x+y)\,dxdy = 1$$

$$\int_0^2\int_0^2 K(x+y)dxdy = 1 \Rightarrow K\int_0^2 \left[\frac{x^2}{2} + xy\right]_0^2 dy = 1$$

$$\Rightarrow K\int_0^2 (2+2y)\,dy = 1$$

$$\Rightarrow K\left[2y + y^2\right]_0^2 = 1$$

$$\Rightarrow K\left[8-0\right] = 1$$

$$\Rightarrow K = \frac{1}{8}$$

(ii). The marginal p.d.f of X is given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \frac{1}{8}\int_0^2 (x+y)dy$$

$$= \frac{1}{8}\left[xy + \frac{y^2}{2}\right]_0^2 = \frac{1+x}{4}$$

$\therefore$ The marginal p.d.f of X is

$$f_X(x) = \begin{cases} \frac{x+1}{4}, & 0 < x < 2 \\ 0, & otherwise \end{cases}$$

The marginal p.d.f of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \frac{1}{8}\int_0^2 (x+y)\,dx$$

$$= \frac{1}{8}\left[\frac{x^2}{2} + yx\right]_0^2$$

$$= \frac{1}{8}\left[2 + 2y\right] = \frac{y+1}{4}$$

$\therefore$The marginal p.d.f of Y is

$$f_Y(y) = \begin{cases} \dfrac{y+1}{4}, & 0 < y < 2 \\ 0, & otherwise \end{cases}$$

(iii). To check whether X and Y are independent or not.

$$f_X(x)\, f_Y(y) = \frac{(x+1)(y+1)}{4 \cdot 4} \neq f_{XY}(x,y)$$

Hence X and Y are not independent.

(iv). Conditional p.d.f $f_{Y/X}\left(y/x\right)$ is given by

$$f_{Y/X}\left(y/x\right) = \frac{f(x,y)}{f_X(x)} = \frac{\frac{1}{8}(x+y)}{\frac{1}{4}(x+1)} = \frac{1}{2}\frac{(x+y)}{(x+1)}$$

$$f_{Y/X}\left(y/x\right) = \frac{1}{2}\left(\frac{x+y}{x+1}\right), \; 0 < x < 2, 0 < y < 2$$

(v) $$P\left(0 < y < \frac{1}{2} \Big/ x = 1\right) = \int_0^{\frac{1}{2}} f_{Y/X}\left(y/x = 1\right) dy$$

$$= \frac{1}{2}\int_0^{\frac{1}{2}} \frac{1+y}{2}\, dy = \frac{5}{32}.$$

Problem 17 a). If X and Y are two random variables having joint probability density function

$$f(x,y) = \begin{cases} \frac{1}{8}(6 - x - y), & 0 < x < 2, 2 < y < 4 \\ 0, & otherwise \end{cases}$$ Find (i) $P(X < 1 \cap Y < 3)$

(ii) $P(X + Y < 3)$ (iii) $P\left(X < 1 \big/ Y < 3\right)$.

b). Three balls are drawn at random without replacement from a box containing 2 white, 3 red and 4 black balls. If X denotes the number of white balls drawn and Y denotes the number of red balls drawn find the joint probability distribution of (X, Y).

Solution:

a).

$$P(X < 1 \cap Y < 3) = \int_{y=-\infty}^{y=3}\int_{x=-\infty}^{x=1} f(x,y)\, dxdy$$

$$= \int_{y=2}^{y=3}\int_{x=0}^{x=1} \frac{1}{8}\left(6 - x - y\right) dxdy$$

$$= \frac{1}{8}\int_{2}^{3}\int_{0}^{1}\left(6 - x - y\right) dxdy$$

$$= \frac{1}{8}\int_{2}^{3}\left[6x - \frac{x^2}{2} - xy\right]_0^1 dy$$

$$= \frac{1}{8}\int_{2}^{3}\left[\frac{11}{2} - y\right]dy = \frac{1}{8}\left[\frac{11y}{2} - \frac{y^2}{2}\right]_2^3$$

$$P\left(X < 1 \cap Y < 3\right) = \frac{3}{8}$$

(ii). $$P\left(X + Y < 3\right) = \int_0^1\int_2^{3-x} \frac{1}{8}\left(6 - x - y\right) dydx$$

$$= \frac{1}{8}\int_0^1\left[6y - xy - \frac{y^2}{2}\right]_2^{3-x} dx$$

$$= \frac{1}{8}\int_0^1\left[6\left(3 - x\right) - x\left(3 - x\right) - \frac{\left(3 - x\right)^2}{2} - \left[12 - 2x - 2\right]\right]dx$$

$$= \frac{1}{8}\int_0^1\left[18 - 6x - 3x + x^2 - \frac{\left(9 + x^2 - 6x\right)}{2} - \left(10 - 2x\right)\right]dx$$

$$= \frac{1}{8}\int_0^1\left[18 - 9x + x^2 - \frac{9}{2} - \frac{x^2}{2} + \frac{6x}{2} - 10 + 2x\right]dx$$

$$= \frac{1}{8}\int_0^1\left[\frac{7}{2} - 4x + \frac{x^2}{2}\right]dx$$

$$= \frac{1}{8}\left[\frac{7x}{2} - \frac{4x^2}{2} + \frac{x^3}{6}\right]_0^1 = \frac{1}{8}\left[\frac{7}{2} - 2 + \frac{1}{6}\right]$$

$$= \frac{1}{8}\left[\frac{21 - 12 + 1}{6}\right] = \frac{1}{8}\left(\frac{10}{6}\right) = \frac{5}{24}.$$

(iii). $$P\left(X < 1 / Y < 3\right) = \frac{P\left(x < 1 \cap y < 3\right)}{P\left(y < 3\right)}$$

The Marginal density function of Y is $f_Y\left(y\right) = \int_0^2 f\left(x, y\right)dx$

$$= \int_0^2 \frac{1}{8}\left(6 - x - y\right)dx$$

$$= \frac{1}{8}\left[6x - \frac{x^2}{2} - yx\right]_0^2$$

$$= \frac{1}{8}\left[12 - 2 - 2y\right]$$

$$= \frac{5-y}{4}, \ 2 < y < 4.$$

$$P\left(X<1 \middle/ Y<3\right) = \frac{\displaystyle\int_{x=0}^{x=1}\int_{y=2}^{y=3} \frac{1}{8}(6 - x - y)\,dx\,dy}{\displaystyle\int_{y=2}^{y=3} f_Y(y)\,dy}$$

$$= \frac{\frac{3}{8}}{\displaystyle\int_2^3 \left(\frac{5-y}{4}\right) dy} = \frac{\frac{3}{8}}{\frac{1}{4}\left[5y - \frac{y^2}{2}\right]_2^3}$$

$$= \frac{3}{8} \times \frac{8}{5} = \frac{3}{5}.$$

b). Let X takes 0, 1, 2 and Y takes 0, 1, 2 and 3.

$P(X = 0, Y = 0) = P$(drawing 3 balls none of which is white or red)

$= P$(all the 3 balls drawn are black)

$$= \frac{4C_3}{9C_3} = \frac{4 \times 3 \times 2 \times 1}{9 \times 8 \times 7} = \frac{1}{21}.$$

$P(X = 0, Y = 1) = P$(drawing 1 red ball and 2 black balls)

$$= \frac{3C_1 \times 4C_2}{9C_3} = \frac{3}{14}$$

$P(X = 0, Y = 2) = P$(drawing 2 red balls and 1 black ball)

$$= \frac{3C_2 \times 4C_1}{9C_3} = \frac{3 \times 2 \times 4 \times 3}{9 \times 8 \times 7} = \frac{1}{7}.$$

$P(X = 0, Y = 3) = P$(all the three balls drawn are red and no white ball)

$$= \frac{3C_3}{9C_3} = \frac{1}{84}$$

$P(X = 1, Y = 0) = P$(drawing 1White and no red ball)

$$= \frac{2C_1 \times 4C_2}{9C_3} = \frac{\frac{2 \times 4 \times 3}{1 \times 2}}{\frac{9 \times 8 \times 7}{1 \times 2 \times 3}}$$

$$= \frac{12 \times 1 \times 2 \times 3}{9 \times 8 \times 7} = \frac{1}{7}$$

$P(X = 1, Y = 1) = P($ drawing 1White and 1 red ball)

$$= \frac{2C_1 \times 3C_1}{9C_3} = \frac{2 \times 3}{\frac{9 \times 8 \times 7}{1 \times 2 \times 3}} = \frac{2}{7}$$

$P(X = 1, Y = 2) = P($ drawing 1White and 2 red ball)

$$= \frac{2C_1 \times 3C_2}{9C_3} = \frac{2 \times 3 \times 2}{\frac{9 \times 8 \times 7}{1 \times 2 \times 3}} = \frac{1}{14}$$

$P(X = 1, Y = 3) = 0$ (Since only three balls are drawn)

$P(X = 2, Y = 0) = P($ drawing 2 white balls and no red balls)

$$= \frac{2C_2 \times 4C_1}{9C_3} = \frac{1}{21}$$

$P(X = 2, Y = 1) = P($ drawing 2 white balls and no red balls)

$$= \frac{2C_2 \times 3C_1}{9C_3} = \frac{1}{28}$$

$P(X = 2, Y = 2) = 0$

$P(X = 2, Y = 3) = 0$

The joint probability distribution of (X, Y) may be represented as

X \ Y	0	1	2	3
0	$\frac{1}{21}$	$\frac{3}{14}$	$\frac{1}{7}$	$\frac{1}{84}$
1	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{1}{14}$	0
2	$\frac{1}{21}$	$\frac{1}{28}$	0	0

Problem 18.a). Two fair dice are tossed simultaneously. Let X denotes the number on the first die and Y denotes the number on the second die. Find the following probabilities.

(i) $P(X + Y) = 8$, (ii) $P(X + Y \geq 8)$, (iii) $P(X = Y)$ and (iv) $P\left(X + Y = 6 / Y = 4\right)$.

b) The joint probability mass function of a bivariate discrete random variable (X, Y) in given by the table.

Y \ X	1	2	3
1	0.1	0.1	0.2
2	0.2	0.3	0.1

Find

i. The marginal probability mass function of X and Y .
ii. The conditional distribution of X given $Y=1$.
iii. $P(X+Y<4)$

Solution:

a). Two fair dice are thrown simultaneously

$$S=\left\{\begin{matrix}(1,1)(1,2)\ldots(1,6)\\(2,1)(2,2)\ldots(2,6)\\ \cdot \quad \cdot \quad \ldots \quad \cdot\\(6,1)(6,2)\ldots(6,6)\end{matrix}\right\},\quad n(S)=36$$

Let X denotes the number on the first die and Y denotes the number on the second die.

Joint probability density function of (X,Y) is $P(X=x,Y=y)=\frac{1}{36}$ for

$x=1,2,3,4,5,6$ and $y=1,2,3,4,5,6$

(i) $X+Y=\{$ the events that the no is equal to 8 $\}$

$=\{(2,6),(3,5),(4,4),(5,3),(6,2)\}$

$$P(X+Y=8)=P(X=2,Y=6)+P(X=3,Y=5)+P(X=4,Y=4)$$
$$+P(X=5,Y=3)+P(X=6,Y=2)$$
$$=\frac{1}{36}+\frac{1}{36}+\frac{1}{36}+\frac{1}{36}+\frac{1}{36}=\frac{5}{36}$$

(ii) $P(X+Y\geq 8)$

$$X+Y=\left\{\begin{matrix}(2,6)\\(3,5),(3,6)\\(4,4),(4,5),(4,6)\\(5,3),(5,4),(5,5),(5,6)\\(6,2),(6,3),(6,4),(6,5),(6,6)\end{matrix}\right\}$$

$$\therefore P(X+Y\geq 8)=P(X+Y=8)+P(X+Y=9)+P(X+Y=10)$$
$$+P(X+Y=11)+P(X+Y=12)$$
$$=\frac{5}{36}+\frac{4}{36}+\frac{3}{36}+\frac{2}{36}+\frac{1}{36}=\frac{15}{36}=\frac{5}{12}$$

(iii) $P(X=Y)$

$$P(X=Y)=P(X=1,Y=1)+P(X=2,Y=2)+\ldots\ldots+P(X=6,Y=6)$$
$$=\frac{1}{36}+\frac{1}{36}+\ldots\ldots\ldots+\frac{1}{36}=\frac{6}{36}=\frac{1}{6}$$

(iv) $P\left(X+Y=6\middle/Y=4\right)=\frac{P(X+Y=6\cap Y=4)}{P(Y=4)}$

Now $P(X+Y=6 \cap Y=4) = \frac{1}{36}$

$P(Y=4) = \frac{6}{36}$

$\therefore P\left(X+Y = 6 \big/ Y=4\right) = \dfrac{\frac{1}{36}}{\frac{6}{36}} = \frac{1}{6}$.

b). The joint probability mass function of (X, Y) is

Y \ X	1	2	3	Total
1	0.1	0.1	0.2	0.4
2	0.2	0.3	0.1	0.6
Total	0.3	0.4	0.3	1

From the definition of marginal probability function

$$P_X(x_i) = \sum_{y_j} P_{XY}(x_i, y_j)$$

When $X = 1$,

$$P_X(x_i) = P_{XY}(1,1) + P_{XY}(1,2)$$
$$= 0.1 + 0.2 = 0.3$$

When $X = 2$,

$$P_X(x=2) = P_{XY}(2,1) + P_{XY}(2,2)$$
$$= 0.2 + 0.3 = 0.4$$

When $X = 3$,

$$P_X(x=3) = P_{XY}(3,1) + P_{XY}(3,2)$$
$$= 0.2 + 0.1 = 0.3$$

$\therefore$ The marginal probability mass function of X is

$$P_X(x) = \begin{cases} 0.3 & when\ x = 1 \\ 0.4 & when\ x = 2 \\ 0.3 & when\ x = 3 \end{cases}$$

The marginal probability mass function of Y is given by $P_Y(y_j) = \sum_{x_i} P_{XY}(x_i, y_j)$

When $Y=1$, $P_Y(y=1) = \sum_{x_i=1}^{3} P_{XY}(x_i, 1)$

$$= P_{XY}(1,1) + P_{XY}(2,1) + P_{XY}(3,1)$$
$$= 0.1 + 0.1 + 0.2 = 0.4$$

When $Y = 2$, $P_Y(y=2) = \sum_{x_i=1}^{3} P_{XY}(x_i, 2)$

$$= P_{XY}(1,2) + P_{XY}(2,2) + P_{XY}(3,2)$$
$$= 0.2 + 0.3 + 0.1 = 0.6$$

$\therefore$ Marginal probability mass function of Y is

$$P_Y(y) = \begin{cases} 0.4 & \text{when } y = 1 \\ 0.6 & \text{when } y = 2 \end{cases}$$

(ii) The conditional distribution of X given $Y = 1$ is given by

$$P\left(X = x / Y = 1\right) = \frac{P(X = x \cap Y = 1)}{P(Y = 1)}$$

From the probability mass function of Y, $P(y = 1) = P_y(1) = 0.4$

When $X = 1$, $P\left(X = 1 / Y = 1\right) = \frac{P(X = 1 \cap Y = 1)}{P(Y = 1)}$

$$= \frac{P_{XY}(1,1)}{P_Y(1)} = \frac{0.1}{0.4} = 0.25$$

When $X = 2$, $P\left(X = 2 / Y = 1\right) = \frac{P_{XY}(2,1)}{P_Y(1)} = \frac{0.1}{0.4} = 0.25$

When $X = 3$, $P\left(X = 3 / Y = 1\right) = \frac{P_{XY}(3,1)}{P_Y(1)} = \frac{0.2}{0.4} = 0.5$

(iii). $P(X + Y < 4) = P\{(x, y) / x + y < 4 \text{ Where } x = 1, 2, 3; y = 1, 2\}$

$$= P\{(1,1), (1,2), (2,1)\}$$
$$= P_{XY}(1,1) + P_{XY}(1,2) + P_{XY}(2,1)$$
$$= 0.1 + 0.1 + 0.2 = 0.4$$

Problem 19.a). If X and Y are two random variables having the joint density function $f(x, y) = \frac{1}{27}(x + 2y)$ where x and y can assume only integer values 0, 1 and 2, find the conditional distribution of Y for $X = x$.

b). The joint probability density function of (X, Y) is given by $f_{XY}(x, y) = \begin{cases} xy^2 + \frac{x^2}{8}, & 0 \le x \le 2, \quad 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$. Find (i) $P(X > 1)$, (ii) $P(X < Y)$ and (iii) $P(X + Y \le 1)$

Solution:

a). Given X and Y are two random variables having the joint density function

$$f(x, y) = \frac{1}{27}(x + 2y) \quad ----(1)$$

Where $x = 0, 1, 2$ and $y = 0, 1, 2$

Then the joint probability distribution X and Y becomes as follows

$\text{Mean} = \frac{b+a}{2} = \frac{450+550}{2} = 500$

$\text{Variance} = \frac{(b-a)^2}{12} = \frac{(550-450)^2}{12} = 833.33$

By CLT $S_n = X_1 + X_2 + X_3 + X_4$ follows a normal distribution with $N\left(n\mu, n\sigma^2\right)$

The standard normal variable is given by $Z = \frac{S_n - n\mu}{\sqrt{n\sigma^2}}$

when $S_n = 1900$, $Z = \frac{1900 - 4\times 500}{\sqrt{4\times 833.33}} = -\frac{100}{57.73} = -1.732$

when $S_n = 2100$, $Z = \frac{2100 - 2000}{\sqrt{4\times 833.33}} = \frac{100}{57.73} = 1.732$

$\therefore P\left(1900 \le S_n \le 2100\right) = P\left(-1.732 < z < 1.732\right)$

$= 2\times P\left(0 < z < 1.732\right) = 2\times 0.4582 = 0.9164$

.

b). Given $E\left(X_i\right) = \mu$ and $Var\left(X_i\right) = 1.5$ Let $\bar{X}$ denote the sample mean.

By C.L.T. $\bar{X}$ follows $N\left(\mu, \frac{\sqrt{1.5}}{\sqrt{n}}\right)$

We have to find ' n ' such that $P\left(\mu - 0.5 < \bar{X} < \mu + 0.5\right) \ge 0.95$

i.e. $P\left(-0.5 < \bar{X} - \mu < 0.5\right) \ge .95$

$P\left(\left|\bar{X} - \mu\right| < 0.5\right) \ge .95$

$P\left[\left|\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}\right| < 0.5\right] \ge 0.95$

$P\left[|z| < 0.5\frac{\sqrt{n}}{\sigma}\right] \ge 0.95$

$P\left[|z| < 0.5\frac{\sqrt{n}}{\sqrt{1.5}}\right] \ge 0.95$

ie $P\left(|Z| < 0.4082\sqrt{n}\right) \ge 0.95$

Where ' Z ' is the standard normal variable.

The Last value of ' n ' is obtained from $P\left(|Z| < 0.4082\sqrt{n}\right) = 0.95$

$2P\left(0 < z < 0.4082\sqrt{n}\right) = 0.95$

$\Rightarrow 0.4082\sqrt{n} = 1.96 \Rightarrow n = 23.05$

$\therefore$ The size of the sample must be atleast 24.

UNIT – III RANDOM PROCESSES

PART - A

Problem 1. Define I & II order stationary Process
Solution:
I Order Stationary Process:
A random process is said to be stationary to order one if is first order density function does not change with a shift in time origin.

i.e., $f_X(x_1 : t_1) = f_X(x_1, t_1 + \delta)$ for any time t_1 and any real number δ.

i.e., $E[X(t)] = \overline{X} =$ Constant.

II Order Stationary Process:
A random process is said to be stationary to order two if its second-order density functions does not change with a shift in time origin.

i.e., $f_X(x_1, x_2 : t_1, t_2) = f_X(x_1, x_2 : t_1 + \delta, t_2 + \delta)$ for all t_1, t_2 and δ.

Problem 2. Define wide-sense stationary process
Solution:
A random process $X(t)$ is said to be wide sense stationary (WSS) process if the following conditions are satisfied

(i). $E[X(t)] = \mu$ i.e., mean is a constant

(ii). $R(\tau) = E[X(t_1)X(t_2)]$ i.e., autocorrelation function depends only on the time difference.

Problem 3. Define a strict sense stationary process with an example
Solution:
A random process is called a strongly stationary process (SSS) or strict sense stationary if all its statistical properties are invariant to a shift of time origin.
This means that $X(t)$ and $X(t+\delta)$ have the same statistics for any δ and any t
Example: Bernoulli process is a SSS process

Problem 4. Define n^{th} order stationary process, when will it become a SSS process?
Solution:
A random process $X(t)$ is said to be stationary to order n or n^{th} order stationary if its n^{th} order density function is invariant to a shift of time origin.

i.e., $f_X(x_1, x_2, ..., x_n, t_1, t_2, ..., t_n) = f_X(x_1, x_2, ..., x_n, t_1 + \delta, t_2 + \delta, ..., t_n + \delta)$ for all $t_1, t_2, ..., t_n$ & h.

10. Define ergodic process.
Solution:
A random process $\{X(t)\}$ is said to be ergodic, if its ensemble average are equal to appropriate time averages.

11. Define a Gaussian process.
Solution:
A real valued random process $\{X(t)\}$ is called a Gaussian process or normal process, if the random variables $X(t_1), X(t_2),..., X(t_n)$ are jointly normal for every $n = 1, 2,...$ and for any set of $t_1, t_2,...$

The n^{th} order density of a Gaussian process is given by

$$f(x_1, x_2,..., x_n; t_1, t_2,..., t_n) = \frac{1}{(2\pi)^{n/2}|\Lambda|^{1/2}} \exp\left[-\frac{1}{2|\Lambda|}\sum_{i=1}^{n}\sum_{j=1}^{n}|\Lambda|_{ij}(x_i - \mu_i)(x_j - \mu_j)\right]$$

Where $\mu_i = E\{X(t_i)\}$ and Λ is the n^{th} order square matrix (λ_{ij}), where $\lambda_{ij} = C\{X(t_i), X(t_j)\}$ and $|\Lambda|_{ij}$ = Cofactor of λ_{ij} in $|\Lambda|$.

12. Define a Markov process with an example.
Solution:
If for $t_1 < t_2 < t_3 < ... < t_n < t$,

$$P\{X(t) \le x / X(t_1) = x_1, X(t_2) = x_2,..., X(t_n) = X_n\} = P\{X(t) \le x / X(t_n) = x_n\}$$

then the process $\{X(t)\}$ is called a markov process.
Example: The Poisson process is a Markov Process.

13. Define a Markov chain and give an example.
Solution:
If for all n,

$$P\{X_n = a_n / X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2},...X_0 = a_0\} = P\{X_n = a_n / X_{n-1} = a_{n-1}\},$$

then the process $\{X_n\}$, $n = 0,1,...$ is called a Markov chain.
Example: Poisson Process is a continuous time Markov chain.

Problem 14. What is a stochastic matrix? When is it said to be regular?
Solution: A sequence matrix, in which the sum of all the elements of each row is 1, is called a stochastic matrix. A stochastic matrix P is said to be regular if all the entries of P^m (for some positive integer m) are positive.

Problem 15. If the transition probability matrix of a markov chain is $\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ find the steady-state distribution of the chain.

Solution:

Let $\pi = (\pi_1, \pi_2)$ be the limiting form of the state probability distribution on stationary state distribution of the markov chain.

By the property of π, $\pi P = \pi$

i.e., $(\pi_1, \pi_2)\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = (\pi_1, \pi_2)$

$\frac{1}{2}\pi_2 = \pi_1$ ________ (1)

$\pi_1 + \frac{1}{2}\pi_2 = \pi_2$ ------ (2)

Equation (1) & (2) are one and the same.

Consider (1) or (2) with $\pi_1 + \pi_2 = 1$, since π is a probability distribution.

$\pi_1 + \pi_2 = 1$

Using (1), $\frac{1}{2}\pi_2 + \pi_2 = 1$

$\frac{3\pi_2}{2} = 1$

$\pi_2 = \frac{2}{3}$

$\pi_1 = 1 - \pi_2 = 1 - \frac{2}{3} = \frac{1}{3}$

$\pi_2 = 1 - \pi_1 = 1 - \frac{1}{3} = \frac{2}{3}$

$\therefore \pi_1 = \frac{1}{3}$ & $\pi_2 = \frac{2}{3}$.

PART-B

Problem 16. a). Define a random (stochastic) process. Explain the classification of random process. Give an example to each class.

Solution:

RANDOM PROCESS

A random process is a collection (orensemble) of random variables $\{X(s,t)\}$ that are functions of a real variable, namely time t where $s \in S$ (sample space) and $t \in T$ (Parameter set or index set).

CLASSIFICATION OF RANDOM PROCESS

Depending on the continuous on discrete nature of the state space S and parameter set T, a random process can be classified into four types:

(i). It both T & S are discrete, the random process is called a discrete random sequence.

Example: If X_n represents the outcome of the n^{th} toss of a fair dice, then $\{X_n, n \geq 1\}$ is a discrete random sequence, since $T = \{1,2,3,...\}$ and $S = \{1,2,3,4,5,6\}$.

(ii). If T is discrete and S is continuous, the random process is called a continuous random sequence.

Example: If X_n represents the temperature at the end n^{th} hour of a day, then $\{X_n, 1 \le n \le 24\}$ is a continuous random sequence since temperature can take any value is an interval and hence continuous.

(iii). If T is continuous and S is discrete, the random process is called a discrete random process.

Example: If $X(t)$ represents the number of telephone calls received in the interval $(0,t)$ then $\{X(t)\}$ random process, since $S = \{0,1,2,3,...\}$.

(iv). If both T and S are continuous, the random process is called a continuous random process f

Example: If $X(t)$ represents the maximum temperature at a place in the interval $(0,t)$ $\{X(t)\}$ is a continuous random process.

b). Consider the random process $X(t) = \cos(t+\varphi)$, where φ is uniformly distributed in the interval $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. Check whether the process is stationary or not.

Solution:

Since θ is uniformly distributed in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$f(\theta) = \frac{1}{\pi}, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$E[X(t)] = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} X(t) f(\theta)\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(t+\theta) \cdot \frac{1}{\pi}\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(t+\theta)\, d\theta$$

$$= \frac{1}{\pi}\left[\sin(t+\theta)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{2}{\pi}\cos t \ne \text{Constant.}$$

Since $E[X(t)]$ is a function of t, the random process $\{X(t)\}$ is not a stationary process.

Problem 17. a). Show that the process $\{X(t)\}$ whose probability distribution under certain conditions is given by
$$P\{X(t)=n\}=\begin{cases}\dfrac{(at)^{n-1}}{(1+at)^{n+1}}, & n=1,2,\ldots\\[2mm] \dfrac{at}{1+at}, & n=0\end{cases}$$
is evolutionary.

Solution:

The probability distribution is given by

$X(t)=n$	:	0	1	2	3	$\ldots$
$P(X(t)=n)$	:	$\dfrac{at}{1+at}$	$\dfrac{1}{(1+at)^2}$	$\dfrac{at}{(1+at)^3}$	$\dfrac{(at)^2}{(1+at)^4}$	$\ldots$

$$E[X(t)]=\sum_{n=0}^{\infty} n p_n$$
$$=\frac{1}{(1+at)^2}+\frac{2at}{(1+at)^3}+\frac{3(at)^2}{(1+at)^4}+\cdots$$
$$=\frac{1}{(1+at)^2}\left\{1+2\left(\frac{at}{1+at}\right)+3\left(\frac{at}{1+at}\right)^2+\cdots\right\}$$
$$=\frac{1}{(1+at)^2}\left[1-\frac{at}{1+at}\right]^{-2}$$
$$E[X(t)]=1=\text{Constant}$$
$$E[X^2(t)]=\sum_{n=0}^{\infty} n^2 p_n$$
$$=\sum_{n=1}^{\infty} n^2\frac{(at)^{n-1}}{(1+at)^{n+1}}=\sum_{n=1}^{\infty}\left[n(n+1)-n\right]\frac{(at)^{n-1}}{(1+at)^{n+1}}$$
$$=\frac{1}{(1+at)^2}\left[\sum_{n=1}^{\infty} n(n+1)\left(\frac{at}{1+at}\right)^{n-1}-\sum_{n=1}^{\infty} n\left(\frac{at}{1+at}\right)^{n-1}\right]$$
$$=\frac{1}{(1+at)^2}\left[2\left(1-\frac{at}{1+at}\right)^{-3}-\left(1-\frac{at}{1+at}\right)^{-2}\right]$$
$$E[X^2(t)]=1+2at\neq\text{Constant}$$
$$Var\{X(t)\}=E\left[X^2(t)\right]-\left[E(X(t))\right]^2$$
$$Var\{X(t)\}=2at$$

$\therefore$ The given process $\{X(t)\}$ is evolutionary b). Examine whether the Poisson process $\{X(t)\}$ given by the probability law $P\{X(t)=n\}=\dfrac{e^{-\lambda t}(\lambda t)^n}{n!}$, $n=0,1,2,\ldots$ is evolutionary.

Solution:

$$E[X(t)]=\sum_{n=0}^{\infty} n p_n$$

$$=\sum_{n=0}^{\infty} n\frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

$$=\sum_{n=1}^{\infty}\frac{e^{-\lambda t}(\lambda t)^n}{(n-1)!}$$

$$=(\lambda t)e^{-\lambda t}\sum_{n=1}^{\infty}\frac{(\lambda t)^{n-1}}{(n-1)!}$$

$$=(\lambda t)e^{-\lambda t}\left[1+\frac{\lambda t}{1!}+\frac{(\lambda t)^2}{2!}+\ldots\right]$$

$$=(\lambda t)e^{-\lambda t}e^{\lambda t}$$

$$E[X(t)]=\lambda t$$

$E[X(t)]\neq$ Constant.

Hence the Poisson process $\{X(t)\}$ is evolutionary.

Problem 18. a). Show that the random process $X(t)=A\cos(\omega t+\theta)$ is WSS if A & ω are constants and θ is uniformly distributed random variable in $(0,2\pi)$.

Solution:

Since θ is uniformly distributed random variable in $(0,2\pi)$

$$f(\theta)=\begin{cases}\dfrac{1}{2\pi}, & 0<0<2\pi\\ 0, & elsewhere\end{cases}$$

$$E[X(t)]=\int_0^{2\pi}X(t)f(\theta)d\theta$$

$$=\int_0^{2\pi}\frac{1}{2\pi}A\cos(\omega t+\theta)d\theta$$

$$=\frac{A}{2\pi}\int_0^{2\pi}\cos(\omega t+\theta)d\theta$$

$$=\frac{A}{2\pi}\left[\sin(\omega t+\theta)\right]_0^{2\pi}$$

$$= \frac{A}{2\pi}\left[\sin(\omega t + 2\pi) - \sin(\omega t)\right]$$

$$= \frac{A}{2\pi}\left[\sin(\omega t) - \sin(\omega t)\right] \qquad \left[\because \sin(2\pi + \theta) = \sin\theta\right]$$

$$E\left[X(t)\right] = 0$$

$$R_{XX}(t_1, t_2) = E\left[X(t_1)X(t_2)\right]$$

$$= E\left[A^2 \cos(\omega t_1 + \theta)\cos(\omega t_2 + \theta)\right]$$

$$= A^2 E\left[\frac{\cos(\omega(t_1 + t_2) + 2\theta) + \cos(\omega(t_1 - t_2))}{2}\right]$$

$$= \frac{A^2}{2}\int_0^{2\pi} \frac{1}{2\pi}\left[\cos(\omega(t_1 + t_2) + 2\theta) + \cos(\omega(t_1 - t_2))\right] d\theta$$

$$= \frac{A^2}{4\pi}\left[\frac{\sin\left[\omega(t_1 + t_2) + 2\theta\right]}{2} + \theta \cos\omega t\right]_0^{2\pi}$$

$$= \frac{A^2}{4\pi}\left[2\pi \cos\omega t\right]$$

$$= \frac{A^2}{2}\cos\omega t = \text{a function of time difference}$$

Since $E\left[X(t)\right] = \text{constant}$

$R_{XX}(t_1, t_2) =$ a function of time difference

$\therefore \{X(t)\}$ is a WSS.

b). Given a random variable y with characteristic function $\varphi(\omega) = E\left(e^{i\omega y}\right)$ and a random process define by $X(t) = \cos(\lambda t + y)$, show that $\{X(t)\}$ is stationary in the wide sense if $\varphi(1) = \varphi(2) = 0$.

Solution:

Given $\varnothing(1) = 0$

$$\Rightarrow E\left[\cos y + i\sin y\right] = 0$$

$$\therefore E\left[\cos y\right] = E\left[\sin y\right] = 0$$

Also $\varnothing(2) = 0$

$$\Rightarrow E\left[\cos 2y + i\sin 2y\right] = 0$$

$$\therefore E\left[\cos 2y\right] = E\left[\sin 2y\right] = 0$$

$$E\{X(t)\} = E\left[\cos(\lambda t + y)\right]$$

$$= E\left[\cos\lambda t \cos y - \sin\lambda t \sin y\right]$$

$$= \cos\lambda t \, E[\cos\lambda t] - \sin\lambda t \, E[\sin y] = 0$$

$$R_{XX}(t_1, t_2) = E[X(t_1) X(t_2)]$$

$$= E[\cos(\lambda t_1 + y)\cos(\lambda t_2 + y)]$$

$$= E\left[\frac{\cos(\lambda(t_1 + t_2) + 2y) + \cos(\lambda(t_1 - t_2))}{2}\right]$$

$$= \frac{1}{2} E[\cos(\lambda(t_1 + t_2) + 2y) + \cos(\lambda(t_1 - t_2))]$$

$$= \frac{1}{2} E[\cos\lambda(t_1 + t_2)\cos 2y - \sin\lambda(t_1 + t_2)\sin 2y + \cos(\lambda(t_1 - t_2))]$$

$$= \frac{1}{2}\cos\lambda(t_1 + t_2) E(\cos 2y) - \frac{1}{2}\sin\lambda(t_1 + t_2) E(\sin 2y) + \frac{1}{2}\cos(\lambda(t_1 - t_2))$$

$$= \frac{1}{2}\cos(\lambda(t_1 - t_2)) = \text{a function of time difference.}$$

Since $E[X(t)] = \text{constant}$

$R_{XX}(t_1, t_2) =$ a function of time difference

$\therefore \{X(t)\}$ is stationary in the wide sense .

Problem 19. a). If a random process $\{X(t)\}$ is defined by $\{X(t)\} = \sin(\omega t + Y)$ where Y is uniformly distributed in $(0, 2\pi)$. Show that $\{X(t)\}$ is WSS.

Solution:

Since y is uniformly distributed in $(0, 2\pi)$,

$$f(y) = \frac{1}{2\pi}, \; 0 < y < 2\pi$$

$$E[X(t)] = \int_0^{2\pi} X(t) f(y) dy$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \sin(\omega t + y) dy$$

$$= \frac{1}{2\pi}\left[-\cos(\omega t + y)\right]_0^{2\pi}$$

$$= -\frac{1}{2\pi}\left[\cos(\omega t + 2\pi) - \cos\omega t\right] = 0$$

$$R_{XX}(t_1, t_2) = E[\sin(\omega t_1 + y)\sin(\omega t_2 + y)]$$

$$= E\left[\frac{\cos(\omega(t_1 - t_2)) - \cos(\omega(t_1 + t_2) + 2y)}{2}\right]$$

$$= \frac{1}{2} E[\cos(\omega(t_1 - t_2))] - \frac{1}{2} E[\cos(\omega(t_1 + t_2) + 2y)]$$

$$= \frac{1}{2}\cos\left(\omega\left(t_1 - t_2\right)\right) - \frac{1}{2}\int_0^{2\pi}\cos\left(\omega\left(t_1 + t_2\right) + 2y\right)\frac{1}{2\pi}dy$$

$$= \frac{1}{2}\cos\left(\omega\left(t_1 - t_2\right)\right) - \frac{1}{4\pi}\left[\frac{\sin\left(\omega\left(t_1 + t_2\right) + 2y\right)}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2}\cos\left(\omega\left(t_1 - t_2\right)\right) - \frac{1}{8\pi}\left[\sin\left(\omega\left(t_1 + t_2\right) + 2\pi\right) - \sin\omega\left(t_1 + t_2\right)\right]$$

$$= \frac{1}{2}\cos\left(\omega\left(t_1 - t_2\right)\right)$$ is a function of time difference.

$\therefore \{X(t)\}$ is WSS.

b). Verify whether the sine wave random process $X(t) = Y\sin\omega t$, Y is uniformly distributed in the interval $(-1,1)$ is WSS or not

Solution:

Since y is uniformly distributed in $(-1,1)$,

$$f(y) = \frac{1}{2}, \quad -1 < y < 1$$

$$E\left[X(t)\right] = \int_{-1}^{1} X(t) f(y) dy$$

$$= \int_{-1}^{1} y\sin\omega t \frac{1}{2} dy$$

$$= \frac{\sin\omega t}{2}\int_{-1}^{1} y dy$$

$$= \frac{\sin\omega t}{2}(0) = 0$$

$$R_{XX}\left(t_1, t_2\right) = E\left[X\left(t_1\right)X\left(t_2\right)\right]$$

$$= E\left[y^2\sin\omega t_1 \sin\omega t_2\right]$$

$$= E\left[y^2\frac{\cos\omega\left(t_1 - t_2\right) - \cos w\left(t_1 + t_2\right)}{2}\right]$$

$$= \frac{\cos\omega\left(t_1 - t_2\right) - \cos w\left(t_1 + t_2\right)}{2}E\left(y^2\right)$$

$$= \frac{\cos\omega\left(t_1 - t_2\right) - \cos w\left(t_1 + t_2\right)}{2}\int_{-1}^{1} y^2 f(y) dy$$

$$= \frac{\cos\omega\left(t_1 - t_2\right) - \cos w\left(t_1 + t_2\right)}{2}\frac{1}{2}\int_{-1}^{1} y^2 dy$$

$$= \frac{\cos\omega(t_1-t_2)-\cos w(t_1+t_2)}{4}\left(\frac{y^3}{3}\right)_{-1}^{1}$$

$$= \frac{\cos\omega(t_1-t_2)-\cos w(t_1+t_2)}{4}\left(\frac{2}{3}\right)$$

$$= \frac{\cos\omega(t_1-t_2)-\cos w(t_1+t_2)}{6}$$

$R_{XX}(t_1,t_2) \neq$ a function of time difference alone.
Hence it is not a WSS Process.

Problem 20. a). Show that the process $X(t) = A\cos\lambda t + B\sin\lambda t$ (where A & B are random variables) is WSS, if (i) $E(A) = E(B) = 0$ (ii) $E(A^2) = E(B^2)$ and (iii) $E(AB) = 0$.

Solution:

Given $X(t) = A\cos\lambda t + B\sin\lambda t$, $E(A) = E(B) = 0$, $E(AB) = 0$,

$E(A^2) = E(B^2) = k(say)$

$E[X(t)] = \cos\lambda t\, E(A) + \sin\lambda t\, E(B)$

$E[X(t)] = 0 =$ is a constant. $\because E(A) = E(B) = 0$

$R(t_1,t_2) = E\{X(t_1)X(t_2)\}$

$= E\{(A\cos\lambda t_1 + B\sin\lambda t_1)(A\cos\lambda t_2 + B\sin\lambda t_2)\}$

$= E(A^2)\cos\lambda t_1\cos\lambda t_2 + E(B^2)\sin\lambda t_1\sin\lambda t_2 + E(AB)[\sin\lambda t_1\cos\lambda t_2 + \cos\lambda t_1\sin\lambda t_2]$

$= E(A^2)\cos\lambda t_1\cos\lambda t_2 + E(B^2)\sin\lambda t_1\sin\lambda t_2 + E(AB)\sin\lambda(t_1+t_2)$

$= k(\cos\lambda t_1\cos\lambda t_2 + \sin\lambda t_1\sin\lambda t_2)$

$= k\cos\lambda(t_1-t_2) =$ is a function of time difference.

$\therefore \{X(t)\}$ is WSS.

b). If $X(t) = Y\cos t + Z\sin t$ for all t & where Y & Z are independent binary random variables. Each of which assumes the values -1 & 2 with probabilities $\frac{2}{3}$ & $\frac{1}{3}$ respectively, prove that $\{X(t)\}$ is WSS.

Solution:

Given

$Y = y$	:	-1	2
$P(Y = y)$	:	$\frac{2}{3}$	$\frac{1}{3}$

$$E(Y) = E(Z) = -1\times\frac{2}{3} + 2\times\frac{1}{3} = 0$$

$$E(Y^2) = E(Z^2) = (-1)^2\times\frac{2}{3} + (2)^2\times\frac{1}{3}$$

$$E(Y^2) = E(Z^2) = \frac{2}{3} + \frac{4}{3} = \frac{6}{3} = 2$$

Since Y & Z are independent

$E(YZ) = E(Y)E(Z) = 0$ ---- (1)

Hence $E[X(t)] = E[y\cos t + z\sin t]$

$= E[y]\cos t + E[z]\sin t$

$E[X(t)] = 0 =$ is a constant. $[\because E(y) = E(z) = 0]$

$R_{XX}(t_1,t_2) = E[X(t_1)X(t_2)]$

$= E[(y\cos t_1 + z\sin t_1)(y\cos t_2 + z\sin t_2)]$

$= E[y^2\cos t_1\cos t_2 + yz\cos t_1\sin t_2 + zy\sin t_1\cos t_2 + z^2\sin t_1\sin t_2]$

$= E(y^2)\cos t_1\cos t_2 + E[yz]\cos t_1\sin t_2 + E[zy]\sin t_1\cos t_2 + E[z^2]\sin t_1\sin t_2$

$= E(y^2)\cos t_1\cos t_2 + E(z^2)\sin t_1\sin t_2$

$= 2[\cos t_1\cos t_2 + \sin t_1\sin t_2]$ $[\because E(y^2) = E(z^2) = 2]$

$= 2\cos(t_1 - t_2) =$ is a function of time difference.

$\therefore \{X(t)\}$ is WSS.

Problem 21. a). Check whether the two random process given by $X(t) = A\cos\omega t + B\sin\omega t$ & $Y(t) = B\cos\omega t - A\sin\omega t$. Show that $X(t)$ & $Y(t)$ are jointly WSS if A & B are uncorrelated random variables with zero mean and equal variance random variables are jointly WSS.

Solution:

Given $E(A) = E(B) = 0$

$Var(A) = Var(B) = \sigma^2$

$\therefore E(A^2) = E(B^2) = \sigma^2$

As A & B uncorrelated are $E(AB) = E(A)E(B) = 0$.

$E[X(t)] = E[A\cos\omega t + B\sin\omega t]$

$= E(A)\cos\omega t + E(B)\sin\omega t = 0$

$E[X(t)] = 0 =$ is a constant.

$R_{XX}(t_1,t_2) = E[X(t_1)X(t_2)]$

$$= E\left[\left(A\cos wt_1 + B\sin wt_2\right)\left(A\cos wt_2 + B\sin wt_2\right)\right]$$

$$= E\left[A^2 \cos wt_1 \cos wt_2 + AB\cos wt_1 \sin wt_2 + BA\sin wt_1 \cos wt_2 + B^2 \sin wt_1 \sin wt_2\right]$$
$$= \cos wt_1 \cos wt_2\, E\left[A^2\right] + \cos wt_1 \sin wt_2\, E[AB] + \sin wt_1 \cos wt_2\, E[BA] + \sin wt_1 \sin wt_2\, E\left[B^2\right]$$
$$= \sigma^2 \left[\cos wt_1 \cos wt_2 + \sin wt_1 \sin wt_2\right]$$
$$= \sigma^2 \cos w\left(t_1 - t_2\right) \qquad \left[\because E\left(A^2\right) = E\left(B^2\right) = \sigma^2 \;\&\; E(AB) = E(BA) = 0\right]$$

$R_{XX}\left(t_1, t_2\right)$ =is a function of time difference.

$$E\left[Y(t)\right] = E\left[B\cos\omega t - A\sin\omega t\right]$$
$$= E(B)\cos\omega t - E(A)\sin\omega t = 0$$

$$R_{YY}\left(t_1, t_2\right) = E\left[\left(B\cos\omega t_1 - A\sin\omega t_1\right)\left(B\cos\omega t_2 - A\sin\omega t_2\right)\right]$$
$$= E\left[B^2 \cos\omega t_1 \cos\omega t_2 - BA\cos\omega t_1 \sin\omega t_2 - AB\sin\omega t_1 \cos\omega t_2 + A^2 \sin\omega t_1 \sin\omega t_2\right]$$
$$= E\left(B^2\right)\cos\omega t_1 \cos\omega t_2 - E(BA)\cos\omega t_1 \sin\omega t_2 - E\left(A^2 B\right)\sin\omega t_1 \cos\omega t_2 + E\left(A^2\right)\sin\omega t_1 \sin\omega t_2$$
$$= \sigma^2 \cos\omega\left(t_1 - t_2\right) \qquad \left[\because E\left(A^2\right) = E\left(B^2\right) = \sigma^2 \;\&\; E(AB) = E(BA) = 0\right]$$

$R_{YY}\left(t_1, t_2\right)$=is a function of time difference.

$$R_{XY}\left(t_1, t_2\right) = E\left[X\left(t_1\right) Y\left(t_2\right)\right]$$
$$= E\left[\left(A\cos\omega t_1 + B\sin\omega t_1\right)\left(B\cos\omega t_2 + A\sin\omega t_2\right)\right]$$
$$= E\left[AB\cos\omega t_1 \cos\omega t_2 - A^2 \cos\omega t_1 \sin\omega t_2 + B^2 \sin\omega t_1 \cos\omega t_2 - BA\sin\omega t_1 \sin\omega t_2\right]$$
$$= \sigma^2\left[\sin\omega t_1 \cos\omega t_2 - \cos\omega t_1 \sin\omega t_2\right]$$
$$= \sigma^2 \sin\omega\left(t_1 - t_2\right) \qquad \left[\because E\left(A^2\right) = E\left(B^2\right) = \sigma^2 \;\&\; E(AB) = E(BA) = 0\right]$$

$R_{XY}\left(t_1, t_2\right)$ = is a function of time difference.

Since $\{X(t)\}$ & $\{Y(t)\}$ are individually WSS & also $R_{XY}\left(t_1, t_2\right)$ is a function of time difference.

$\therefore$ The two random process $\{X(t)\}$ & $\{Y(t)\}$ are jointly WSS.

b). Write a note on Binomial process.

Solution:

Binomial Process can be defined as a sequence of partial sums $\{S_n / n = 1, 2, ...\}$ Where $S_n = X_1 + X_2 + ... + X_n$ Where X_i denotes 1 if the trial is success or 0 if the trial is failure.

As an example for a sample function of the binomial random process with $(x_1, x_2, ...) = (1,1,0,0,1,0,1,...)$ is $(s_1, s_2, s_3, ...) = (1,2,2,2,3,3,4,...)$, The process increments by 1 only at the discrete times t_i $t_i = iT$, $i = 1, 2, ...$

Properties

(i). Binomial process is Markovian

(ii). S_n is a binomial random variable so, $P(S_n = m) = nC_m p^m q^{n-m}$, $E[S_n] = np$ & $var[S_n] = np(1-p)$

(iii) The distribution of the number of slots m_i between i^{th} and $(i-1)^{th}$ arrival is geometric with parameter p starts from 0. The random variables $m_i,\ i = 1, 2, \dots$ are mutually independent.

The geometric distribution is given by $p(1-p)^{i-1}, i = 1, 2, \dots$

(iv) The binomial distribution of the process approaches poisson when n is large and p is small.

Problem 22. a). Describe Poisson process & show that the Poisson process is Markovian.

Solution:

If $\{X(t)\}$ represents the number of occurrences of a certain event in $(0, t)$ then the discrete random process $\{X(t)\}$ is called the Poisson process, provided the following postulates are satisfied

(i) $P[1\ occurrence\ in\ (t, t+\Delta t)] = \lambda \Delta t + o(\Delta t)$

(ii) $P[no\ occurrence\ in\ (t, t+\Delta t)] = 1 - \lambda \Delta t + o(\Delta t)$

(iii) $P[2\ or\ more\ occurrences\ in\ (t, t+\Delta t)] = o(\Delta t)$

(iv) $X(t)$ is independent of the number of occurrences of the event in any interval prior and after the interval $(0, t)$.

(v) The probability that the event occurs in a specified number of times $(t_0, t_0 + t)$ depends only on t, but not on t_0.

Consider

$$P[X(t_3) = n_3 / X(t_2) = n_2, X(t_1) = n_1] = \frac{P[X(t_1) = n_1, X(t_2) = n_2, X(t_3) = n_3]}{P[X(t_1) = n_1, X(t_2) = n_2]}$$

$$= \frac{e^{-\lambda t_3} \lambda^{n_3} t_1^{n_1} (t_2 - t_1)^{n_2 - n_1} (t_3 - t_2)^{n_3 - n_2}}{n_1! (n_2 - n_1)! (n_3 - n_2)!}$$

$$= \frac{e^{-\lambda t_2} \lambda^{n_2} t_1^{n_1} (t_2 - t_1)^{n_2 - n_1}}{n_1! (n_2 - n_1)!}$$

$$= \frac{e^{-\lambda (t_3 - t_2)} \lambda^{n_3 - n_2} (t_3 - t_2)^{n_3 - n_2}}{(n_3 - n_2)!}$$

$$P[X(t_3) = n_3 / X(t_2) = n_2, X(t_1) = n_1] = P[X(t_3) = n_3 / X(t_2) = n_2]$$

This means that the conditional probability distribution of $X(t_3)$ given all the past values $X(t_1) = n_1$, $X(t_2) = n_2$ depends only on the most recent values $X(t_2) = n_2$.

UNIT – IV CORRELATION AND SPECTRAL DENSITIES

PART-A

Problem1. Define autocorrelation function and prove that for a WSS process $\{X(t)\}$, $R_{XX}(-\tau) = R_{XX}(\tau)$ 2.State any two properties of an autocorrelation function.

Solution:

Let $\{X(t)\}$ be a random process. Then the auto correlation function of the process $\{X(t)\}$ is the expected value of the product of any two members $X(t_1)$ and $X(t_2)$ of the process and is given by $R_{XX}(t_1,t_2) = E[X(t_1)X(t_2)]$ or $R_{XX}(t,t+\tau) = E[X(t)X(t+\tau)]$

For a WSS process $\{X(t)\}$, $R_{XX}(\tau) = E[X(t)X(t-\tau)]$

$\therefore R_{XX}(-\tau) = E[X(t)X(t+\tau)] = E[X(t+\tau)X(t)] = R(t+\tau-t) = R(\tau)$

Problem 2. State any two properties of an autocorrelation function.

Solution:

The process $\{X(t)\}$ is stationary with autocorrelation function $R(\tau)$ then

(i) $R(\tau)$ is an even function of τ

(ii) $R(\tau)$ is maximum at $\tau = 0$ i.e., $|R(\tau)| \le R(0)$

Problem 3. Given that the autocorrelation function for a stationary ergodic process with no periodic components is $R(\tau) = 25 + \frac{4}{1+6\tau^2}$. Find the mean and variance of the process $\{X(t)\}$.

Solution:

$$\mu_x^2 = \underset{\tau\to\infty}{Lt} R(\tau) = \underset{\tau\to\infty}{Lt}\, 25 + \frac{4}{1+6\tau^2} = 25$$

$\therefore \mu_x = 5$

$E(X^2(t)) = R_{XX}(0) = 25 + 4 = 29$

$Var(X(t)) = E[X^2(t)] - E[X(t)]^2 = 29 - 25 = 4$

Problem 4. Find the mean and variance of the stationary process $\{X(t)\}$ whose autocorrelation function $R(\tau) = \frac{25\tau^2 + 36}{6.25\tau^2 + 4}$.

Solution:

$$R(\tau) = \frac{25\tau^2 + 36}{6.25\tau^2 + 4} = \frac{25 + \frac{36}{\tau^2}}{6.25 + \frac{4}{\tau^2}}$$

$$\mu_x^2 = \underset{\tau \to \infty}{Lt} R(\tau) = \frac{25}{6.25} = \frac{2500}{625} = 4$$

$$\mu_x = 2$$

$$E\left[X^2(t)\right] = R_{XX}(0) = \frac{36}{4} = 9$$

$$Var\left[X(t)\right] = \left\{E\left[X(t)\right]\right\}^2 - E\left[X(t)\right]^2 = 9 - 4 = 5$$

Problem 5. Define cross-correlation function and mention two properties.
Solution:

The cross-correlation of the two process $\{X(t)\}$ and $\{Y(t)\}$ is defined by $R_{XY}(t_1, t_2) = E\left[X(t_1)Y(t_2)\right]$

Properties: The process $\{X(t)\}$ and $\{Y(t)\}$ are jointly wide-sense stationary with the cross-correlation function $R_{XY}(\tau)$ then

(i) $R_{YX}(\tau) = R_{XY}(-\tau)$

(ii) $\left|R_{XY}(\tau)\right| \le \sqrt{R_{XX}(0)\, R_{YY}(0)}$

Problem 6. Find the mean – square value of the process $\{X(t)\}$ if its autocorrelation function is given by $R(\tau) = e^{-\tau^2/2}$.

Solution:

Mean-Square value $= E\left[X^2(t)\right] = R_{XX}(0) = \left(e^{-\frac{\tau^2}{2}}\right)_{\tau=0} = 1$

Problem 7. Define the power spectral density function (or spectral density or power spectrum) of a stationary process?
Solution:

If $\{X(t)\}$ is a stationary process (either in the strict sense or wide sense with auto correlation function $R(\tau)$, then the Fourier transform of $R(\tau)$ is called the power spectral density function of $\{X(t)\}$ and denoted by $S_{XX}(\omega)$ or $S(\omega)$ or $S_X(\omega)$

Thus $S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau$

Problem 8. State any two properties of the power spectral density function.
Solution:
(i). The spectral density function of a real random process is an even function.

(ii). The spectral density of a process $\{X(t)\}$, real or complex, is a real function of ω and non-negative.

Problem 9. State Wiener-Khinchine Theorem.

Solution:

If $X_T(\omega)$ is the Fourier transform of the truncated random process defined as

$$X_T(t) = \begin{cases} X(t) & \text{for } |t| \le T \\ 0 & \text{for } |t| > T \end{cases}$$

Where $\{X(t)\}$ is a real WSS Process with power spectral density function $S(\omega)$, then

$$S(\omega) = \underset{T\to\infty}{Lt}\left[\frac{1}{2T} E\left\{|X_T(\omega)|^2\right\}\right]$$

Problem 10. If $R(\tau) = e^{-2\lambda|\tau|}$ is the auto Correlation function of a random process $\{X(t)\}$, obtain the spectral density of $\{X(t)\}$.

Solution:

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} e^{-2\lambda|\tau|}(\cos\omega\tau - i\sin\omega\tau)d\tau$$

$$= 2\int_0^{\infty} e^{-2\lambda\tau}\cos\omega\tau\, d\tau$$

$$= \left[\frac{2e^{-2\lambda\tau}}{4\lambda^2+\omega^2}(-2\lambda\cos\omega\tau + \omega\sin\omega\tau)\right]_0^{\infty}$$

$$S(\omega) = \frac{4\lambda}{4\lambda^2+\omega^2}$$

Problem 11. The Power spectral density of a random process $\{X(t)\}$ is given by

$$S_{XX}(\omega) = \begin{cases} \pi, & |\omega| < 1 \\ 0, & \text{elsewhere} \end{cases}$$ Find its autocorrelation function.

Solution:

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega) e^{i\omega\tau} d\omega$$

$$= \frac{1}{2\pi}\int_{-1}^{1} \pi e^{i\omega\tau} d\omega$$

$$= \frac{1}{2}\left[\frac{e^{i\omega\tau}}{i\tau}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\frac{e^{i\tau} - e^{-i\tau}}{i\tau}\right]$$

$$= \frac{1}{\tau}\left[\frac{e^{i\tau}-e^{-i\tau}}{2i}\right] = \frac{\sin\tau}{\tau}$$

Problem 12. Define cross-Spectral density.

Solution:

The process $\{X(t)\}$ and $\{Y(t)\}$ are jointly wide-sense stationary with the cross-correlation function $R_{XY}(\tau)$, then the Fourier transform of $R_{XY}(T)$ is called the cross spectral density function of $\{X(t)\}$ and $\{Y(t)\}$ denoted as $S_{XY}(\omega)$

Thus $S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-i\omega\tau} d\tau$

Problem 13. Find the auto correlation function of a stationary process whose power spectral density function is given by $s(\omega) = \begin{cases} \omega^2 & for \ |\omega| \le 1 \\ 0 & for \ |\omega| > 1 \end{cases}$.

Solution:

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega) e^{i\omega\tau} d\omega$$

$$= \frac{1}{2\pi}\int_{-1}^{1} \omega^2 (\cos\omega\tau + i\sin\omega\tau) d\omega$$

$$= \int_{-1}^{1} \omega^2 .\cos\omega\tau \, d\omega = \frac{1}{\pi}\left[\omega^2\left(\frac{\sin\omega\tau}{\tau}\right) - 2\omega\left(\frac{-\cos\omega\tau}{\tau^2}\right) + 2\left(\frac{-\sin\omega\tau}{\tau^3}\right)\right]_0^1$$

$$R(\tau) = \frac{1}{\pi}\left[\frac{\sin\tau}{\tau} + \frac{2\cos\tau}{\tau^2} - \frac{2\sin\tau}{\tau^3}\right]$$

Problem 14. Given the power spectral density : $S_{xx}(\omega) = \frac{1}{4+\omega^2}$, find the average power of the process.

Solution:

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega) e^{i\omega\tau} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[\frac{1}{4+\omega^2}\right] e^{i\omega\tau} d\omega$$

Hence the average power of the processes is given by

$$E\left[X^2(t)\right] = R(0)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{d\omega}{4+\omega^2}$$

$$= \frac{1}{2\pi} 2\int_0^{\infty} \frac{d\omega}{2^2+\omega^2}$$

$$= \frac{1}{\pi}\left[\frac{1}{2}\tan^{-1}\left(\frac{\omega}{2}\right)\right]_0^{\infty}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{4} - 0\right] = \frac{1}{4}.$$

Problem 15. Find the power spectral density of a random signal with autocorrelation function $e^{-\lambda|\tau|}$.

Solution:

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} e^{-\lambda|\tau|}(\cos\omega\tau - i\sin\omega\tau) d\tau$$

$$= 2\int_0^{\infty} e^{-\lambda|\tau|}\cos\omega\tau\, d\tau$$

$$= 2\left[\frac{e^{-\lambda\tau}}{\lambda^2+\omega^2}(-\lambda\cos\omega\tau + \omega\sin\omega\tau)\right]_0^{\infty}$$

$$= 2\left[0 - \frac{1}{\lambda^2+\omega^2}(-\lambda)\right] = \frac{2\lambda}{\lambda^2+\omega^2}$$

PART-B

Problem 16. a). If $\{X(t)\}$ is a W.S.S. process with autocorrelation function $R_{XX}(\tau)$ and if $Y(t) = X(t+a) - X(t-a)$. Show that $R_{YY}(\tau) = 2R_{XX}(\tau) - R_{XX}(\tau+2a) - R_{XX}(\tau-2a)$.

Solution:

$$R_{yy}(\tau) = E[y(t)y(t+\tau)]$$

$$= E\{[X(t+a) - X(t-a)][X(t+\tau+a) - X(t+\tau-a)]\}$$

$$= E[X(t+a)X(t+\tau+a)] - E[X(t+a)X(t+\tau-a)]$$

$$- E[X(t-a)X(t+\tau+a)] + E[X(t-a)X(t+\tau-a)]$$

$$= R_{xx}(\tau) - E[X(t+a)X(t+a+\tau-2a)]$$

$$- E[X(t-a)X(t-a+\tau+2a)] + R_{xx}(\tau)$$

$$= 2R_{xx}(\tau) - R_{xx}(\tau-2a) - R_{xx}(\tau+2a)$$

b). Assume a random signal $Y(t) = X(t) + X(t-a)$ where $X(t)$ is a random signal and 'a' is a constant. Find $R_{YY}(\tau)$.

Solution:

$$R_{YY}(\tau) = E[Y(t)Y(t+\tau)]$$

$$= E\{[X(t) + X(t-a)][X(t+\tau) + X(t+\tau-a)]\}$$

$$= E[X(t)X(t+\tau)] + E[X(t)X(t+\tau-a)]$$
$$+ E[X(t-a)X(t+\tau)] + E[X(t-a)X(t+\tau-a)]$$
$$= R_{xx}(\tau) + R_{xx}(\tau+a) + R_{xx}(\tau-a) + R_{xx}(\tau)$$
$$R_{yy}(\tau) = 2R_{xx}(\tau) + R_{xx}(\tau+a) + R_{xx}(\tau-a)$$

Problem 17. a). If $\{X(t)\}$ and $\{Y(t)\}$ are independent WSS Processes with zero means, find the autocorrelation function of $\{Z(t)\}$, when (*i*) $Z(t) = a + bX(t) + CY(t)$, (*ii*) $Z(t) = aX(t)Y(t)$.

Solution:

Given $E[X(t)] = 0$, $E[Y(t)] = 0$ -------- (1)

$\therefore \{X(t)\}$ and $\{Y(t)\}$ are independent

$E\{X(t)Y(t)\} = E[X(t)Y(t)] = 0$ --------(2)

(i). $R_{ZZ}(\tau) = E[Z(t)Z(t+\tau)]$

$$= E\{[a + bX(t) + cY(t)][a + bX(t+\tau) + cY(t+\tau)]\}$$
$$= E\{a^2 + ab\,X(t+\tau) + acY(t+\tau) + baX(t) + b^2X(t)X(t+\tau)$$
$$+ bc\,X(t)Y(t+\tau) + caY(t) + cbY(t)X(t+\tau) + c^2y(t)y(t+\tau)\}$$
$$= E(a^2) + abE[X(t+\tau)] + acE[Y(t+\tau)] + baE[X(t)] + b\;E[X(t)X(t+\tau)]$$
$$+ bc\,E[X(t)y(t+\tau)] + caE[Y(t)] + cbE[Y(t)X(t+\tau)] + c^2E[Y(t)Y(t+\tau)]$$
$$= a^2 + b^2R_{XX}(\tau) + c^2R_{YY}(\tau)$$

$R_{ZZ}(\tau) = E[Z(t)Z(t+\tau)]$

$$= E[aX(t)Y(t)aX(t+\tau)Y(t+\tau)]$$
$$= E[a^2X(t)X(t+\tau)Y(t)Y(t+\tau)]$$
$$= a^2E[X(t)X(t+\tau)]E[Y(t)Y(t+\tau)]$$
$$R_{ZZ}(\tau) = a^2R_{XX}(\tau)R_{YY}(\tau)$$

b). If $\{X(t)\}$ is a random process with mean 3 and autocorrelation $R_{xx}(\tau) = 9 + 4e^{-0.2|\tau|}$. Determine the mean, variance and covariance of the random variables $Y = X(5)$ and $Z = X(8)$.

Solution:

Given $Y = X(5)$ & $Z = X(8)$, $E[X(t)] = 3$ ———— (1)

Mean of $Y = E[Y] = E[X(5)] = 3$

Mean of $Z = E[Z] = E[X(8)] = 3$

We know that

$Var(Y) = E(Y^2) - [E(Y)]^2$

$E(Y^2) = E(X^2(5))$

But $E[X^2(t)] = R_{XX}(0)$

$= 9 + 4e^{-0.2|0|}$

$= 9 + 4 = 13$

Thus $Var(Y) = 13 - (3)^2 = 13 - 9 = 4$

$Var(Z) = E(Z^2) - [E(Z)]^2$

$E[Z^2] = E[X^2(8)] \quad [\because Z = X(8)]$

$= R_{XX}(0)$

$= 9 + 4 = 13$

Hence $Var(Z) = 13 - (3^2) = 13 - 9 = 4$

$E[YZ] = R(5,8) = 9 + 4e^{-0.2|5-8|} \quad \left(\because R(t_1, t_2) = 9 + 4e^{-0.2|t_1 - t_2|}\right)$

$= 9 + 4e^{-0.6}$

$\text{Covariance} = R(t_1, t_2) - E[X(t_1)]\, E[X(t_2)]$

$= R(5,8) - E[5]\, E[8]$

$= 9 + 4e^{-0.6} - (3 \times 3) = 4e^{-0.6} = 2.195$

Problem 18. a). The autocorrelation function for a stationary process is given by $R_{xx}(\tau) = 9 + 2e^{-|\tau|}$. Find the mean value of the random variable $Y = \int_0^2 X(t)\,dt$ and variance of $X(t)$.

Solution:

Given $R_{xx}(\tau) = 9 + 2e^{-|\tau|}$

Mean of $X(t)$ is given by

$\overline{X}^2 = [E\,X(t)]^2 = \underset{|\tau| \to \infty}{Lt} R_{xx}(\tau)$

$= \underset{|\tau| \to \infty}{Lt} \left(9 + 2e^{-|\tau|}\right)$

$\overline{X}^2 = 9$

$\overline{X} = 3$

Also $E[X^2(t)] = R_{XX}(0) = 9 + 2e^{|0|} = 9 + 2 = 11$

$Var\{X(t)\} = E[X^2(t)] - [E(X(t))]^2$

$= 11 - 3^2 = 11 - 9 = 2$

Mean of $Y(t) = E[Y(t)]$

$$= E\left[\int_0^2 X(t)\,dt\right]$$

$$= \int_0^2 E\left[X(t)\right] dt$$

$$= \int_0^2 3\,dt = 3(t)_0^2 = 6$$

$\therefore E\left[Y(t)\right] = 6$

b). Find the mean and autocorrelation function of a semi random telegraph signal process.

Solution:

Semi random telegraph signal process.

If $N(t)$ represents the number of occurrences of a specified event in $(0,t)$ and $X(t) = (-1)^{N(t)}$, then $\{X(t)\}$ is called the semi random signal process and $N(t)$ is a poisson process with rate λ.

By the above definition $X(t)$ can take the values $=1$ and -1 only

$$P\left[X(t) = 1\right] = P\left[N(t)\, is\ even\right]$$

$$= \sum_{K=even} \frac{e^{-\lambda t}(\lambda t)^k}{K!}$$

$$= e^{-\lambda t}\left[1 + \frac{(\lambda t)^2}{2} + \ldots\right]$$

$$P\left[X(t) = 1\right] = e^{-\lambda t} cosh\lambda t$$

$$P\left[X(t) = -1\right] = P\left[N(t)\, is\ odd\right]$$

$$= \sum_{K=odd} \frac{e^{-\lambda t}(\lambda t)^k}{K!}$$

$$= e^{-\lambda t}\left[\lambda t + \frac{(\lambda t)^3}{3!} + \ldots\right]$$

$$= e^{-\lambda t} sinh\lambda t$$

$$Mean\{X(t)\} = \sum_{K=-1,1} K\,P\left(X(t) = K\right)$$

$$= 1 \times e^{-\lambda t} cosh\lambda t + (-1) \times e^{-\lambda t} sinh\lambda t$$

$$= e^{-\lambda t}\left[cosh\lambda t - sinh\lambda t\right]$$

$$= e^{-2\lambda t}\quad\left[cosh\lambda t - sinh\lambda t = e^{-\lambda t}\right]$$

$$R(\tau) = E\left[X(t)X(t+\tau)\right]$$

$$= 1 \times P\left[X(t)X(t+\tau) = 1\right] + -1 \times P\left[X(t)X(t+\tau) = -1\right]$$

$$= \sum_{n=even} e^{\lambda t} \frac{(\lambda\tau)^n}{n!} - \sum_{n=odd} e^{-\lambda} \frac{(\lambda\tau)^n}{n!}$$
$$= e^{-\lambda\tau} cosh\lambda\tau - e^{\lambda\tau} sinh\lambda\tau$$
$$= e^{-\lambda\tau}\left[cosh\lambda\tau - sinh\lambda\tau\right]$$
$$= e^{-\lambda\tau} e^{-\lambda\tau}$$
$$R(\tau) = e^{-2\lambda\tau}$$

Problem 19. a). Find Given the power spectral density of a continuous process as

$$S_{XX}(\omega) = \frac{\omega^2+9}{\omega^4+5\omega^2+4}$$

find the mean square value of the process.

Solution:

We know that mean square value of $\{X(t)\}$

$$= E\{X^2(t)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\,d\omega$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\omega^2+9}{(\omega^4+5\omega^2+4)}\,d\omega$$
$$= \frac{1}{2\pi} 2\int_{0}^{\infty} \frac{\omega^2+9}{\omega^4+5\omega^2+4}\,d\omega$$
$$= \frac{1}{\pi}\int_{0}^{\infty} \frac{\omega^2+9}{\omega^4+\omega^2+4\omega^2+4}\,d\omega$$
$$= \frac{1}{\pi}\int_{0}^{\infty} \frac{\omega^2+9}{\omega^2(\omega^2+1)+4(\omega^2+1)}\,d\omega$$

i.e., $$E\{X^2(t)\} = \frac{1}{\pi}\int_{0}^{\infty} \frac{\omega^2+9}{(\omega^2+4)(\omega^2+1)}\,d\omega$$

let $\omega^2 = u$

$\therefore$ We have $$\frac{\omega^2+9}{(\omega^2+4)(\omega^2+1)} = \frac{u+9}{(u+4)(u+1)}$$

$$= \frac{\frac{-4+9}{-4+1}}{u+4} + \frac{\frac{-1+9}{-1+4}}{u+1} = -\frac{5}{3(u+4)} + \frac{8}{3(u+1)}$$ ….Partial fractions

i.e., $$\frac{\omega^2+9}{(\omega^2+4)(\omega^2+1)} = -\frac{5}{3(\omega^2+4)} + \frac{8}{3(\omega^2+1)}$$

$\therefore$From (1),

$$E\{X^2(t)\} = \frac{1}{\pi}\int_{0}^{\infty} \frac{1}{3}\left[\frac{-5}{(\omega^2+4)} + \frac{8}{(\omega^2+1)}\right]d\omega$$

$$= \frac{1}{3\pi}\left[-5.\frac{1}{2}\tan^{-1}\frac{\omega}{2} + 8\tan^{-1}\omega\right]_0^{\infty}$$

$$= \frac{1}{3\pi}\left[-\frac{5}{2}\left(\frac{\pi}{2}\right) + 8\left(\frac{\pi}{2}\right) - 0\right]$$

$$= \frac{1}{3\pi}\cdot\frac{\pi}{2}\left(-\frac{5}{2} + 8\right) = \frac{1}{6}\left(\frac{11}{2}\right)$$

$$E\left\{X^2(t)\right\} = \frac{11}{12}.$$

b). If the $2n$ random variables A_r and B_r are uncorrelated with zero mean and $E\left(A_r^2\right) = E\left(B_r^2\right) = \sigma_r^2$. Find the mean and autocorrelation of the process $X(t) = \sum_{r=1}^{n} A_r\cos\omega_r t + B_r\sin\omega_r t$.

Solution:

Given $E(A_r) = E(B_r) = 0$ & $E\left(A_r^2\right) = E\left(B_r^2\right) = \sigma_r^2$

Mean: $E\left[X(t)\right] = E\left[\sum_{r=1}^{n} A_r\cos\omega_r t + B_r\sin\omega_r t\right]$

$$= \sum_{r=1}^{n}\left[E(A_r)\cos\omega_r t + E(B_r)\sin\omega_r t\right]$$

$E\left[X(t)\right] = 0 \qquad \because E(A_r) = E(B_r) = 0$

Autocorrelation function:

$$R(\tau) = E\left[X(t)X(t+\tau)\right]$$

$$= E\left\{\sum_{r=1}^{n}\sum_{s=1}^{n}\left(A_r\cos\omega_r t + B_r\sin\omega_r t\right)\left(A_s\cos\omega_s(t+\tau) + B_s\sin\omega_s(t+\tau)\right)\right\}$$

Given $2n$ random variables A_r and B_r are uncorrelated

$E[A_r A_s], E[A_r B_s], E[B_r A_s], E[B_r, B_s]$ are all zero for $r \neq s$

$$= \sum_{r=1}^{n} E\left(A_r^2\right)\cos\omega_r t\cos\omega_r(t+\tau) + E\left(B_r^2\right)\sin\omega_r t\sin\omega_r(t+\tau)$$

$$= \sum_{r=1}^{n}\sigma_r^2\cos\omega_r(t - t - \tau)$$

$$= \sum_{r=1}^{n}\sigma_r^2\cos\omega_r(-\tau)$$

$$R(\tau) = \sum_{r=1}^{n}\sigma_r^2\cos\omega_r\tau$$

Problem 20. a). If $\{X(t)\}$ is a WSS process with autocorrelation $R(\tau) = Ae^{-\alpha|\tau|}$, determine the second – order moment of the random variable $X(8) - X(5)$.

Solution:

UNIT – V LINEAR SYSTEMS WITH RANDOM INPUTS

PART - A

Problem 1. If the system function of a convolution type of linear system is given by $h(t) = \begin{cases} \frac{1}{2a} & \text{for } |t| \le a \\ 0 & \text{for } |t| > a \end{cases}$ find the relation between power spectrum density function of the input and output processes.

Solution:

$$H(\omega) = \int_{-a}^{a} h(t) e^{-i\omega t} dt = \frac{\sin a\omega}{a\omega}$$

We know that $S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$

$$\Rightarrow S_{YY}(\omega) = \frac{\sin^2 a\omega}{a^2\omega^2} S_{XX}(\omega).$$

Problem 2. Give an example of cross-spectral density.

Solution:

The cross-spectral density of two processes $X(t)$ and $Y(t)$ is given by $S_{XY}(\omega) = \begin{cases} p + iq\omega, & \text{if } |\omega| < 1 \\ 0, & \text{otherwise} \end{cases}$

Problem 3. If a random process $X(t)$ is defined as $X(t) = \begin{cases} A, & 0 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}$, where A is a random variable uniformly distributed from $-\theta$ to θ. Prove that autocorrelation function of $X(t)$ is $\frac{\theta^2}{3}$.

Solution:

$$R_{XX}(t, t+\tau) = E[X(t).X(t+\tau)]$$
$$= E[A^2] \quad [\because X(t) \text{ is constant}]$$

But A is uniform in $(-\theta, \theta)$

$$\therefore f(\theta) = \frac{1}{2\theta}, -\theta < a < \theta$$

$$\therefore R_{XX}(t, t+\tau) = \int_{-\theta}^{\theta} a^2 f(a)\, da$$
$$= \int_{-\theta}^{\theta} a^2 . \frac{1}{2\theta} d\theta = \frac{1}{2\theta}\left[\frac{a^3}{3}\right]_{-\theta}^{\theta}$$

$$= \frac{1}{6\theta}\left[\theta^3 - (-\theta)^3\right] = \frac{1}{6\theta}.2\theta^3 = \frac{\theta^2}{3}$$

Problem 4. Check whether $\frac{1}{1+9\tau^2}$ is a valid autocorrelation function of a random process.

Solution: Given $R(\tau) = \frac{1}{1+9\tau^2}$

$$\therefore R(-\tau) = \frac{1}{1+9(-\tau^2)} = \frac{1}{1+9\tau^2} = R(\tau)$$

$\therefore R(\tau)$ is an even function. So it can be the autocorrelation function of a random process.

Problem 5. Find the mean square value of the process X(t) whose power density spectrum is $\frac{4}{4+\omega^2}$.

Solution:

Given $S_{XX}(\omega) = \frac{4}{4+\omega^2}$

Then $R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{i\tau\omega}\,d\omega$

Mean square value of the process is $E\left[X^2(t)\right] = R_{XX}(0)$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4}{4+\omega^2}\,d\omega$$

$$= \frac{4}{\pi}\int_{0}^{\infty} \frac{1}{4+\omega^2}\,d\omega \qquad \left[\frac{1}{4+\omega^2} \textit{ is even}\right]$$

$$= \frac{4}{\pi}.\frac{1}{2}\left[\tan^{-1}\frac{\omega}{2}\right]_0^{\infty} = \frac{2}{\pi}\left(\tan^{-1}\infty - \tan^{-1}0\right)$$

$$= \frac{2}{\pi}.\frac{\pi}{2} = 1$$

Problem 6. A Circuit has an impulse response given by $h(t) = \begin{cases} \frac{1}{T}; & 0 \le t \le T \\ 0; & elsewhere \end{cases}$ find the relation between the power spectral density functions of the input and output processes.

Solution:

$$H(\omega) = \int_0^T h(t)e^{-i\omega t}\,dt$$

$$= \int_0^T \frac{1}{T} e^{-i\omega t} dt$$

$$= \frac{1}{T}\left[\frac{e^{-i\omega t}}{-i\omega}\right]_0^T$$

$$= \frac{1}{T}\left[\frac{e^{-i\omega T} - 1}{-i\omega}\right]$$

$$= \frac{\left(1 - e^{-i\omega T}\right)}{Ti\omega}$$

$$S_{YY}(\omega) = \left|H(\omega)\right|^2 S_{XX}(\omega)$$

$$= \frac{\left|1 - e^{-i\omega T}\right|^2}{\omega^2 T^2} S_{XX}(\omega)$$

Problem 7. Describe a linear system.

Solution:

Given two stochastic process $\{X_1(t)\}$ and $\{X_2(t)\}$, we say that L is a linear transformation if

$$L\left[a_1 X_1(t) + a_2 X_2(t)\right] = a_1 L\left[X_1(t)\right] + a_2 L\left[X_2(t)\right]$$

Problem 8. Given an example of a linear system.

Solution:

Consider the system f with output $tx(t)$ for an input signal $x(t)$.

i.e. $y(t) = f\left[X(t)\right] = t\,x(t)$

Then the system is linear.

For any two inputs $x_1(t), x_2(t)$ the outputs are $tx_1(t)$ and $tx_2(t)$ Now

$$f\left[a_1 x_1(t) + a_2 x_2(t)\right] = t\left[a_1 x_1(t) + a_2 x_2(t)\right]$$

$$= a_1 t x_1(t) + a_2 t x_2(t)$$

$$= a_1 f\left(x_1(t)\right) + a_2 f\left(x_2(t)\right)$$

$\therefore$ the system is linear.

Problem 9. Define a system, when it is called a linear system?

Solution:

Mathematically, a system is a functional relation between input $x(t)$ and output $y(t)$.

Symbolically, $y(t) = f\left[x(t)\right], -\infty < t < \infty$..

The system is said to be linear if for any two inputs $x_1(t)$ and $x_2(t)$ and constants a_1, a_2, $f\left[a_1 x_1(t) + a_2 x_2(t)\right] = a_1 f\left[x_1(t)\right] + a_2 f\left[x_2(t)\right]$.

Problem 10. State the properties of a linear system.

Solution:

Let $X_1(t)$ and $X_2(t)$ be any two processes and a and b be two constants.

If L is a linear filter then

$$L\left[a_1 x_1(t) + a_2 x_2(t)\right] = a_1 L\left[x_1(t)\right] + a_2 L\left[x_2(t)\right].$$

Problem 11. Describe a linear system with an random input.

Solution:

We assume that $X(t)$ represents a sample function of a random process $\{X(t)\}$, the system produces an output or response $Y(t)$ and the ensemble of the output functions forms a random process $\{Y(t)\}$. The process $\{Y(t)\}$ can be considered as the output of the system or transformation f with $\{X(t)\}$ as the input the system is completely specified by the operator f.

Problem 12. State the convolution form of the output of linear time invariant system.

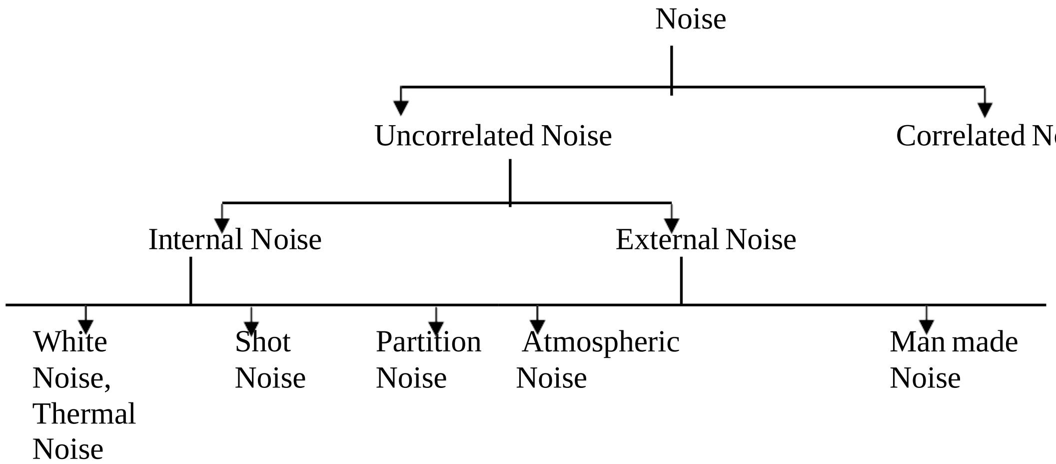
Solution:

If $X(t)$ is the input and $h(t)$ be the system weighting function and $Y(t)$ is the output, then $Y(t) = h(t) * X(t) = \int_{-\infty}^{\infty} h(u) X(t-u)\,du$

Problem 13. Write a note on noise in communication system.

Solution:

The term noise is used to designate unwanted signals that tend to disturb the transmission and processing of signal in communication systems and over which we have incomplete control.



Problem 14. Define band-limited white noise.

Solution:

Noise with non-zero and constant density over a finite frequency band is called band-limit white noise i.e.,

$$S_{NN}(\omega) = \begin{cases} \dfrac{N_0}{2}, & |\omega| \le \omega_B \\ 0, & \text{otherwise} \end{cases}$$

Problem 15. Define (a) Thermal Noise (b) White Noise.

Solution:

(a) Thermal Noise: This noise is due to the random motion of free electrons in a conducting medium such as a resistor.

(or)

Thermal noise is the name given to the electrical noise arising from the random motion of electrons in a conductor.

(b)White Noise(or) Gaussian Noise: The noise analysis of communication systems is based on an idealized form of noise called White Noise.

PART-B

Problem 16. A random process $X(t)$ is the input to a linear system whose impulse response is $h(t) = 2e^{-t}, t \ge 0$. If the autocorrelation function of the process is $R_{XX}(\tau) = e^{-2|\tau|}$, find the power spectral density of the output process $Y(t)$.

Solution:

Given $X(t)$ is the input process to the linear system with impulse response $h(t) = 2e^{-t}, t \ge 0$

So the transfer function of the linear system is its Fourier transform

$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt$$

$$= \int_{0}^{\infty} 2e^{-t} e^{-i\omega t} dt \qquad \left[\because 2e^{-t}, t \ge 0\right]$$

$$= 2\int_{0}^{\infty} e^{-(1+i\omega)t} dt$$

$$= 2\left[\frac{e^{-(1+i\omega)t}}{-(1+i\omega)}\right]_0^{\infty}$$

$$= \frac{-2}{1+i\omega}[0-1] = \frac{2}{1+i\omega}$$

Given $R_{XX}(\tau) = e^{-2|\tau|}$

$\therefore$ the spectral density of the input is

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} e^{-2|\tau|} e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{0} e^{2\tau} e^{-i\omega\tau} d\tau + \int_{0}^{\infty} e^{-2\tau} e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{0} e^{(2-i\omega)\tau} d\tau + \int_{0}^{\infty} e^{-(2+i\omega)\tau} d\tau$$

$$= \left[\frac{e^{(2-i\omega)\tau}}{2-i\omega}\right]_{-\infty}^{0} + \left[\frac{e^{-(2-i\omega)\tau}}{-(2+i\omega)}\right]_{0}^{\infty}$$

$$= \frac{1}{2-i\omega}[1-0] - \frac{1}{2+i\omega}[0-1]$$

$$= \frac{1}{2-i\omega} + \frac{1}{2+i\omega}$$

$$= \frac{2+i\omega+2-i\omega}{(2+i\omega)(2-i\omega)} = \frac{4}{4+\omega^2}$$

We know the power spectral density of the output process $Y(t)$ is given by

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$

$$= \left|\frac{2}{1+i\omega}\right|^2 \frac{4}{4+\omega^2}$$

$$= \frac{4}{(1+\omega^2)} \frac{4}{4+\omega^2}$$

$$= \frac{16}{(1+\omega^2)(4+\omega^2)}$$

Problem 17. If $Y(t) = A\cos(\omega_0 t + \theta) + N(t)$, where A is a constant, θ is a random variable with uniform distribution in $(-\pi, \pi)$ and $N(t)$ is a band-limited Gaussian white noise with a power spectral density $S_{NN}(\omega) = \begin{cases} \frac{N_0}{2}, & for \ |\omega - \omega_0| < \omega_B \\ 0, & elsewhere \end{cases}$. Find the power spectral density of $Y(t)$. Assume that $N(t)$ and θ are independent.

Solution:

Given $Y(t) = A\cos(\omega_0 t + \theta) + N(t)$

$N(t)$ is a band-limited Gaussian white noise process with power spectral density

$$S_{NN}(\omega) = \frac{N_0}{2}, \ |\omega - \omega_0| < \omega_B \quad ie. \ \omega_0 - \omega_B < \omega < \omega_0 + \omega_B$$

Required $S_{YY}(\omega) = \int_{-\infty}^{\infty} R_{YY}(\tau) e^{-i\omega\tau} d\tau$

Now $R_{YY}(\tau) = E[Y(t)Y(t+\tau)]$

$$= E\{[A\cos(\omega_0 t + \theta) + N(t)][A\cos(\omega_0 t + \omega_0\tau + \theta) + N(t+\tau)]\}$$

$$= E\left\{ \begin{array}{l} A^2\cos(\omega_0 t+\theta)\cos(\omega_0 t+\omega_0\tau+\theta)+N(t)N(t+\tau)+A\cos(\omega_0 t+\theta)N(t+\tau) \\ +A\cos(\omega_0 t+\omega_0\tau+\theta)N(t) \end{array} \right\}$$

$$= A^2E\left[\cos(\omega_0 t+\theta).\cos(\omega_0 t+\omega_0\tau+\theta)\right]+E\left[N(t)N(t+\tau)\right]$$

$$+AE\left[\cos(\omega_0 t+\theta)\right]E\left[N(t+\tau)\right]$$

$$+AE\left[\cos(\omega_0 t+\omega_0+\theta)\right]E\left[N(t)\right] \qquad [\because \theta \text{ and } N(t) \text{ are independent}]$$

$$= \frac{A^2}{2}\left\{E\left[\cos(2\omega_0 t+\omega_0 t+2\theta)\right]+\cos\omega_0\tau\right\}+R_{NN}(\tau)$$

$$+AE\left[\cos(\omega_0 t+\theta)\right]E\left[N(t+\tau)\right] \qquad +AE\left[\cos(\omega_0 t+\omega_0\tau+2\theta)\right]E\left[N(t)\right]\}$$

Since θ is uniformly distributed in $(-\pi,\pi)$ the pdf of θ is $f(\theta)=\frac{1}{2\pi}, -\pi<\theta<\pi$

$$\therefore E\left[\cos(\omega_0 t+\theta)\right]=\int_{-\pi}^{\pi}\cos(\omega_0 t+\theta)f(\theta)d\theta$$

$$= \int_{-\pi}^{\pi}\left[\cos\omega_0 t.\cos\theta-\sin\omega_0 t.\sin\theta\right]\frac{1}{2\pi}d\theta$$

$$= \frac{1}{2\pi}\left[\cos\omega_0 t\int_{-\pi}^{\pi}\cos\theta d\theta-\sin\omega_0 t\int_{-\pi}^{\pi}\sin\theta d\theta\right]$$

$$= \frac{1}{2\pi}\left[\cos\omega_0 t\left[\sin\theta\right]_{-\pi}^{\pi}-\sin\omega_0 t.0\right]=0$$

Similarly $E\left[\cos(2\omega_0 t+\omega_0\tau+2\theta)\right]=\int_{-\pi}^{\pi}\cos(2\omega_0 t+\omega_0\tau+2\theta)\frac{1}{2\pi}d\theta$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\cos(2\omega_0 t+\omega_0\tau)\cos 2\theta-\sin(2\omega_0 t+\omega_0\tau)\sin 2\theta\right]d\theta$$

$$= \frac{1}{2\pi}\left\{\cos(2\omega_0 t+\omega_0\tau)\int_{-\pi}^{\pi}\cos 2\theta d\theta-\sin(2\omega_0 t+\omega_0\tau)\int_{-\pi}^{\pi}\sin 2\theta d\theta\right\}$$

$$= \frac{1}{2\pi}\left\{\cos(2\omega_0 t+\omega_0\tau).\left[\frac{\sin 2\theta}{2}\right]_{-\pi}^{\pi}-\sin(2\omega_0 t+\omega_0\tau).0\right\}=0$$

$$\therefore R_{YY}(\tau)=\frac{A^2}{2}\cos\omega_0\tau+R_{NN}(\tau)$$

$$\therefore S_{YY}(\omega)=\int_{-\infty}^{\infty}\left[\frac{A^2}{2}\cos\omega_0\tau+R_{NN}(\tau)\right]e^{-i\omega\tau}d\tau$$

$$= \frac{A^2}{2}\int_{-\infty}^{\infty}\cos\omega_0\tau e^{-i\omega\tau}d\tau+\int_{-\infty}^{\infty}R_{NN}(\tau)e^{-i\omega\tau}d\tau$$

$$= \frac{\pi A^2}{2}\left\{\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right\}+S_{NN}(\omega)$$

$$= \frac{\pi A^2}{2}\left\{\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right\}+\frac{N_0}{2},\ \omega_0-\omega_B<\omega<\omega_0+\omega_B$$

Problem 18. Consider a Gaussian white noise of zero mean and power spectral density $\frac{N_0}{2}$ applied to a low pass RC filter whose transfer function is $H(f)=\frac{1}{1+i2\pi fRC}$. Find the autocorrelation function.

Solution:

The transfer function of a RC circuit is given. We know if $X(t)$ is the input process and $Y(t)$ is the output process of a linear system, then the relation between their spectral densities is $S_{YY}(\omega)=|H(\omega)|^2 S_{XX}(\omega)$

The given transfer function is in terms of frequency f

$$S_{YY}(f)=|H(f)|^2 S_{XX}(f)$$

$$S_{YY}(f)=\frac{1}{1+4\pi^2 f^2R^2C^2}\frac{N_0}{2}$$

$$\therefore R_{YY}(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_{YY}(\omega)e^{i\omega\tau}d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i2\pi\tau f}}{1+4\pi^2 f^2R^2C^2}\frac{N_0}{2}\,df$$

$$=\frac{N_0}{4\pi}\int_{-\infty}^{\infty}\frac{e^{i(2\pi\tau)f}}{4\pi^2R^2C^2\left(\frac{1}{4\pi^2R^2C^2}+f^2\right)}df$$

$$=\frac{N_0}{16\pi^3R^2C^2}\int_{-\infty}^{\infty}\frac{e^{i(2\pi\tau)f}}{\left(\frac{1}{2\pi RC}\right)^2+f^2}df$$

We know from contour integration $\int_{-\infty}^{\infty}\frac{e^{imx}}{a^2+x^2}dx=\frac{\pi}{a}e^{-|m|a}$

$$\therefore R_{YY}(\tau)=\frac{N_0}{16\pi^3R^2C^2}\frac{\pi}{\frac{1}{2\pi RC}}e^{-\frac{\tau}{2\pi RC}}$$

$$=\frac{N_0}{16\pi^3R^2C^2}2\pi^2RC\,e^{-\frac{\tau}{2\pi RC}}$$

$$=\frac{N_0}{8\pi RC}e^{-\frac{\tau}{2\pi RC}}$$

Problem 19. A wide-sense stationary noise process N(t) has an autocorrelation function $R_{NN}(\tau)=Pe^{-3|\tau|}$ where P is a constant Find its power spectrum

Solution:

Given the autocorrelation function of the noise process $N(t)$ is $R_{NN}(\tau) = Pe^{-3|\tau|}$

$$\therefore S_{NN}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} P e^{-3|\tau|} e^{-i\omega\tau} d\tau$$

$$= P\left\{ \int_{-\infty}^{0} e^{3\tau} e^{-i\omega\tau} d\tau + \int_{0}^{\infty} e^{-3\tau} e^{-i\omega\tau} d\tau \right\}$$

$$= P\left\{ \int_{-\infty}^{0} e^{(3-i\omega)\tau} d\tau + \int_{0}^{\infty} e^{-(3+i\omega)\tau} d\tau \right\}$$

$$= P\left\{ \left[\frac{e^{(3-i\omega)\tau}}{3-i\omega} \right]_{-\infty}^{0} + \left[\frac{e^{-(3+i\omega)\tau}}{-(3+i\omega)} \right]_{0}^{\infty} \right\}$$

$$= P\left\{ \frac{1}{3-i\omega}(1-0) - \frac{1}{3+i\omega}(0-1) \right\}$$

$$= P\left\{ \frac{1}{3-i\omega} + \frac{1}{3+i\omega} \right\}$$

$$= P\left\{ \frac{3+i\omega+3-i\omega}{(3-i\omega)(3+i\omega)} \right\} = \frac{6P}{9+\omega^2}$$

Problem 20. A wide sense stationary process X(t) is the input to a linear system with impulse response $h(t) = 2e^{-7t}, t \ge 0$. If the autocorrelation function of X(t) is $R_{XX}(\tau) = e^{-4|\tau|}$, find the power spectral density of the output process Y(t).

Solution:

Given X(t) is a WSS process which is the input to a linear system and so the output process Y(t) is also a WSS process (by property autocorrelation function)

Further the spectral relationship is $S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$

Where $S_{XX}(\omega)$ = Fourier transform of $R_{XX}(\tau)$

$$= \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} e^{-4|\tau|} e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{0} e^{4\tau} e^{-i\omega\tau} d\tau + \int_{0}^{\infty} e^{-4\tau} e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{0} e^{\tau(4-i\omega)} d\tau + \int_{0}^{\infty} e^{-\tau(4+i\omega)} d\tau$$

$\Rightarrow R_{XY}(\tau) = R_{XX}(\tau) * h(\tau)$

(b). Now $R_{YX}(\tau) = R_{XY}(-\tau)$

$= R_{XY}(-\tau) * h(-\tau)$ [*from* (*i*)]

$= R_{XX}(\tau) * h(-\tau)$ [Since $R_{XX}(\tau)$ is an even function of τ]

(c). $R_{YY}(t, t-\tau) = E[Y(t)Y(t-\tau)]$

$$= E\left[\int_{-\infty}^{\infty} h(u)X(t-u)du\,Y(t-\tau)\right]$$

$$= E\left[\int_{-\infty}^{\infty} X(t-u)Y(t-\tau)h(u)du\right]$$

$$= \int_{-\infty}^{\infty} E[X(t-u)Y(t-\tau)]h(u)du = \int_{-\infty}^{\infty} R_{XY}(\tau-u)h(u)du$$

It is a function of τ only and it is true for any τ.

$\therefore R_{YY}(\tau) = R_{XY}(\tau) * h(\tau)$

Problem 33. Prove that the mean of the output of a linear system is given by $\mu_Y = H(0)\mu_X$, where $X(t)$ is WSS.

Solution:

We know that the input $X(t)$, output $Y(t)$ relationship of a linear system can expressed as a convolution $Y(t) = h(t) * X(t)$

$$= \int_{-\infty}^{\infty} h(u)(t-u)du$$

Where $h(t)$ is the unit impulse response of the system.

$\therefore$the mean of the output is

$$E[Y(t)] = E\left[\int_{-\infty}^{\infty} h(u)X(t-u)du\right] = \int_{-\infty}^{\infty} h(u)E[X(t-u)]du$$

Since X(t) is WSS, $E[X(t)] = \mu_X$ is a constant for any t.

$E[X(t-u)] = \mu_X$

$$\therefore E[Y(t)] = \int_{-\infty}^{\infty} h(u)\mu_X\,du = \mu_X \int_{-\infty}^{\infty} h(u)du$$

We know $H(\omega)$ is the Fourier transform of $h(t)$.

i.e. $H(\omega) = \int_{-\infty}^{\infty} h(t)dt$

put $\omega = 0 \therefore H(0) = \int_{-\infty}^{\infty} h(t)dt = \int_{-\infty}^{\infty} h(u)du$

$\therefore E[Y(t)] = \mu_X H(0)$.